Essen (video)
27/4/2021

Lecture 3

Serre weights and diagrams for $GL_2(\mathbb{Q}_{p^f})$

[a survey of B.- Paškūnas]

[talk not intended for experts]

p = prime number, $K = \mathbb{Q}_{p^f}$ $(f \geq 1)$, $\mathbb{F}$ = coef. field = finite extension of $\mathbb{F}_p$

I mainly survey (old) results of Paškūnas and myself (quoted BP) on certain admis. smooth repres. of $GL_2(K)$ over $\mathbb{F}$ with prescribed $GL_2(\mathcal{O}_K)$-socle = Serre weights of Buzzard Diamond-Jarvis (quoted BDJ).

(1) Serre weights of BDJ (generic case)
(2) Weight cycling and multiplicity 1
(3) Diagrams.

[I will also mention more recent results.]

(1) Serre weights of BDJ

Take your favorite global setting among the following 3:

(i) F = totally real number field, $D/_F$ = quaternion algebra wich is split ($\simeq M_2$) at places above p and split at one infinite place (and only one).

(ii) idem but $D/_F$ is definite at all infinite places

(iii) F^+ = tot. real number field, F = tot. imaginary quadratic extension of F^+ such that any $v|p$ in F^+ splits in F, $G/_{F^+}$ = unitary group wich is GL_2 at places above p and compact at all infinite places.

Rk: In (iii) there are 2 places of F above any $v|p$ in F^+: I ignore this in the sequel.

I fix $v | p$ unramified in $\begin{array}{c} F \\ | \\ F^+ \end{array}$ and let K be the correspon- (2)
-ding completion. I also fix $\bar{r} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_2(\mathbb{F})$ which
is continuous absolutely irreducible.

For every compact open subgroup U^v in $(D \otimes_F \mathbb{A}_F^{\infty, v})^\times$ or $G(\mathbb{A}_{F^+}^{\infty, v})$

finite adeles outside v

I define:

$$\pi_v[\bar{r}] := \begin{cases} \text{(i) } \mathrm{Hom}_{\mathrm{Gal}(\bar{F}/F)}\left(\bar{r}, \varinjlim_{U_v} H^1_{et}(X_{U^v U_v} \times_F \bar{F}, \mathbb{F})\right) \text{ where } U_v = \text{c.o. s/g} \\ \text{in } (D \otimes_F F_v)^\times \simeq GL_2(K) \text{ and } X_{U^v U_v} = \text{(smooth projective)} \\ \text{Shimura curve}/F \text{ with level } U^v U_v \\ \\ \text{(ii) } \varinjlim_{U_v} \left\{ f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times / U^v U_v \longrightarrow \mathbb{F} \right\} [\mathfrak{m}_{\bar{r}}] \\ \\ \text{(iii) } \varinjlim_{U_v} \left\{ f : G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty) / U^v U_v \longrightarrow \mathbb{F} \right\} [\mathfrak{m}_{\bar{r}}] \end{cases}$$

$\mathfrak{m}_{\bar{r}}$: max. ideal (in spherical Hecke alg.) associated to $\bar{r}$

$\pi_v[\bar{r}]$ = smooth admissible represent$^{\underline{o}}$ of $GL_2(K)$ over $\mathbb{F}$ (in cases (ii) and (iii) action is $(g \cdot f)(\cdot) = f(\cdot\, g)$).

let $K_1 := 1 + pM_2(\mathcal{O}_K) \subseteq GL_2(\mathcal{O}_K)$, $\pi_v[\bar{r}]$ admissible $\Longrightarrow$ $\pi_v[\bar{r}]^{K_1}$ is finite dim. $/\mathbb{F}$ $\Rightarrow$ $\mathrm{socle}_{GL_2(\mathcal{O}_K)}(\pi_v[\bar{r}])$ is also f.d. $/\mathbb{F}$ (recall socle = maximal semi-simple subrepresent$^{\underline{o}}$).

$\Rightarrow$ $\mathrm{socle}_{GL_2(\mathcal{O}_K)}(\pi_v[\bar{r}])$ is the direct sum of finitely many Serre weights of $GL_2(\mathcal{O}_K)$ (or of $GL_2(\mathbb{F}_q) \simeq GL_2(\mathcal{O}_K)/K_1$).

We assume $\pi_v[\bar r] \neq 0$ from now on ($\Leftrightarrow \bar r$ modular).

BDJ gives the list of these Serre weights up to multiplicity [which I describe below]

I let $\bar\rho := \bar r_v = \bar r|_{\mathrm{Gal}(\bar K/K)}$ and assume $\bar\rho$ is generic in the following sense:

[according to normaliz. one may have to take $\bar r_v(1)$ or $\bar r_v^\vee$]

- if $\bar\rho$ is reducible, then $\bar\rho|_{I_K} \simeq \begin{pmatrix} \omega_f^{r_0+1+p(r_1+1)+\cdots+p^{f-1}(r_{f-1}+1)} & * \\ 0 & 1 \end{pmatrix} \otimes \eta|_{I_K}$ (inertia: I_K)

for some char. η of $G_K := \mathrm{Gal}(\bar K/K)$ and some $r_j \in \{0, \dots, p-3\}$ such that $(r_0, \dots, r_{f-1}) \neq (0, \dots, 0), (p-3, \dots, p-3)$

- if $\bar\rho$ is irreducible, then $\bar\rho|_{I_K} \simeq \begin{pmatrix} \omega_{2f}^{r_0+1+\cdots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & \omega_{2f}^{q(\text{same})} \end{pmatrix} \otimes \eta|_{I_K}$

for some η, r_j s.t. $r_0 \in \{1, \dots, p-2\}$, $r_j \in \{0, \dots, p-3\}$, $i \neq 0$.

ω_f, ω_{2f} := Serre's fundamental characters of level f, $2f$ (one needs to fix an embedding $\mathbb{F}_{p^{2f}} \hookrightarrow \mathbb{F}$, which I do).

<u>Notation for Serre weights (BP)</u>: $s_j \in \{0, \dots, p-1\}$, $0 \le j \le f-1$, $\theta : \mathcal{O}_K^\times \to \mathbb{F}^\times$

$$(s_0, \dots, s_{f-1}) \otimes \theta := \mathrm{Sym}^{s_0}(\mathbb{F}^2) \otimes_{\mathbb{F}} (\mathrm{Sym}^{s_1}(\mathbb{F}^2))^{\mathrm{Fr}} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} (\mathrm{Sym}^{s_{f-1}}(\mathbb{F}^2))^{\mathrm{Fr}^{f-1}} \otimes \theta \circ \det$$

where $(\ )^{\mathrm{Fr}^j}$ means $GL_2(\mathbb{F}_q)$ acts via $\mathbb{F}_q \xrightarrow{x \mapsto x^{p^j}} \mathbb{F}_q \hookrightarrow \mathbb{F}$. (fixed embedding)

<u>Set of Serre weights associated to $\bar\rho|_{I_K}$ (BDJ)</u>:

- if $\bar\rho$ is reducible, then $W(\bar\rho)$:= set of $(s_0, \dots, s_{f-1}) \otimes \theta$ such that there is $J \subseteq \{0, \dots, f-1\}$ with

$$\bar\rho|_{I_K} \simeq \begin{pmatrix} \omega_f^{\sum_{j\notin J}(s_j+1)p^j} & * \\ 0 & \omega_f^{\sum_{j\in J}(s_j+1)p^j} \end{pmatrix} \otimes \theta$$ where the extension [use LCFT]

between the above 2 characters is <u>Fontaine-Laffaille</u> (i.e. "crystalline mod. p")

- if $\bar\rho$ is irreducible, then $W(\bar\rho) := $ set of $(s_0, \dots, s_{f-1}) \otimes \theta$ such that $$\bar\rho|_{I_K} \simeq \begin{pmatrix} \omega_{2f}^{\sum_{j\notin J}(s_j+1)p^j + q\sum_{j\in J}(s_j+1)p^j} & 0 \\ 0 & \omega_{2f}^{q(\text{same})} \end{pmatrix} \otimes \theta.$$

<u>Example for $f=2$ and $\bar\rho$ reducible</u>

One has
$$\begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & * \\ 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} \omega_2^{r_0+2} & * \\ 0 & \omega_2^{p(p-1-r_1)} \end{pmatrix} \otimes \omega_2^{p-1+pr_1}$$
always FL}
$$\simeq \begin{pmatrix} \omega_2^{p(r_1+2)} & * \\ 0 & \omega_2^{p-1-r_0} \end{pmatrix} \otimes \omega_2^{r_0+p(p-1)}$$
can be FL}
$$\simeq \begin{pmatrix} 1 & * \\ 0 & \omega_2^{p-2-r_0+p(p-2-r_1)} \end{pmatrix} \otimes \omega_2^{r_0+1+p(r_1+1)}.$$
FL iff $* = 0$}

$\Rightarrow W(\bar\rho^{ss}) = \{(r_0, r_1),\ (r_0+1, p-2-r_1) \otimes \det^{p-1+pr_1},\ (p-2-r_0, r_1+1) \otimes \det^{r_0+p(p-1)},$
$(p-3-r_0, p-3-r_1) \otimes \det^{r_0+1+p(r_1+1)}\}$

$|W(\bar\rho)| \in \{1, 2, 4\}$ with $4 \Leftrightarrow \bar\rho = \bar\rho^{ss}$; if $\bar\rho$ non split, then
$W(\bar\rho) = \{(r_0, r_1)\}$ or $\{(r_0, r_1), (r_0+1, p-2-r_1) \otimes \det^{p-1+pr_1}\}$ or
$\{(r_0, r_1), (p-2-r_0, r_1+1) \otimes \det^{r_0+p(p-1)}\}$
(skip)

<u>I now sum up several facts about $W(\bar\rho)$:</u>

- $(r_0, \dots, r_{f-1})$ is always in $W(\bar\rho)$

• $(p-3-r_0, \cdots, p-3-r_{f-1}) \otimes \det^{2(r_j+1)p^j}$ is in $W(\bar{\rho})$ iff $\bar{\rho}$ reduc. split ⑤

• if $\bar{\rho}$ is semi-simple (generic), then all Serre weights in $W(\bar{\rho})$ have the following form: the sequences $p-2-, \cdots, +1$ can "loop"

$$(\cdots, r_{i-1}, p-2-r_i, p-3-r_{i+1}, \cdots, p-3-r_{i+l}, r_{i+l}+1, r_{i+l+1}, \cdots, r_{j-1}, p-2-r_j, p-3-, \cdots, +1, \cdots) \otimes \det$$

with $\begin{cases} p-3-r_0 \\ r_0+1 \end{cases}$ replaced by $\begin{cases} p-1-r_0 \\ r_0-1 \end{cases}$ if $\bar{\rho}$ is irreducible, and $*$ being given by a unique formula in terms of the r_j and of the sequences $p-2-, p-3-, \cdots, +1$; in particular $|W(\bar{\rho})| = 2^f$; we define the <u>length</u> of a Serre weight in $W(\bar{\rho})$ as the sum of the length of all the sequences $p-2-, \cdots\cdots, +1$, i.e. (r_i) has length 0, $(p-2-r_0, r_1+1)$ has length 1, $(p-2-r_0, p-3-r_1, r_2+1, r_3)$, $(p-2-r_0, r_1+1, p-2-r_2, r_3+1)$ have length 2, etc.

• if $\bar{\rho}$ is reducible nonsplit, [there is a unique Serre weight of maximal length, and all other Serre weights have their $p-2-, \cdots, +1$ sequences "contained" in the maximal one]; in particular $|W(\bar{\rho})| = 2^d$, $0 \leq d \leq f-1$.
skip

I now go back to $\pi_v[\bar{r}]$:

<u>Theorem A</u> (Barnet-Lamb, Gee, Geraghty, Kisin, Liu, Savitt, ...)
Assume $p>5$ is unramified in F + $\bar{r}|_{G_{F(\sqrt[p]{1})}}$ is irreducible + further technical assumptions on global situation (iii) (F/F^+ unramified at finite places, G quasi-split at finite places, $\bar{r}$ has split ramification, etc.). Then $W(\bar{\rho})$ is exactly the set of Serre weights up to multiplicity in $\mathrm{soc}_{GL_2(\mathcal{O}_K)}(\pi[\bar{r}])$.

② Weight cycling and multiplicity 1

$I := \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix}$ = Iwahori $\quad I_1 := \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix}$ = pro-p Iwahori

$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \begin{vmatrix} I_1 \\ I \end{vmatrix} = \begin{vmatrix} I_1 \\ I \end{vmatrix} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ respects $\pi_v[\bar{r}]^{I_1}$ (inside $\pi[\bar{r}]$)

Weight cycling is a way to relate the various $\underset{\uparrow \text{ has dim} = 1}{\sigma^{I_1}}$ $(\hookrightarrow \pi[\bar{r}]^{I_1})$ for $\sigma \in W(\bar{\rho})$ using $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. (idea initially due to Buzzard for $K = \mathbb{Q}_p$)

If $\chi : I \to \mathbb{F}^\times$, then $\chi : I \twoheadrightarrow \underset{\begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix}}{\overset{}{I/I_1}} \to \mathbb{F}^\times \Rightarrow \chi\begin{pmatrix} a & b \\ pc & d \end{pmatrix} = \chi_1(a)\chi_2(d)$ for $\chi_i : \mathcal{O}_K^\times \to \mathbb{F}^\times$;

write $\chi = \chi_1 \otimes \chi_2$ and define

$$\chi^s := \chi_2 \otimes \chi_1 = \chi\left(\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}^{-1}\right).$$

Now, let $\sigma \hookrightarrow \pi_v[\bar{r}]$ $(\sigma \in W(\bar{\rho}))$, χ the action of I on σ^{I_1}, then I acts on $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\sigma^{I_1}$ by χ^s.

<u>Frobenius reciprocity</u>: one has a $GL_2(\mathcal{O}_K)$-equivariant surjection:

$$\mathrm{ind}_I^{GL_2(\mathcal{O}_K)} \chi^s \twoheadrightarrow \left\langle GL_2(\mathcal{O}_K) \cdot \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \sigma^{I_1} \right\rangle \quad (\hookrightarrow \pi[\bar{r}])$$

$$\cong$$

$$\mathrm{ind}_{B(\mathbb{F}_q)}^{GL_2(\mathbb{F}_q)} \chi^s = \text{Principal Series for } GL_2(\mathbb{F}_q)$$

What is the quotient $\langle \cdots \rangle$ of $\mathrm{ind}\,\chi^s$?

<u>Lemma 1 (BP)</u>: There is a (unique) smallest non zero quotient of $\mathrm{ind}\,\chi^s$ with socle a Serre weight in $W(\bar{\rho})$.

Guess in BP: this is this quotient

Denote by $\delta(\sigma)$ the Serre weight of the Lemma.

One can study $\sigma \rightsquigarrow \delta(\sigma) \rightsquigarrow \delta^2(\sigma) \rightsquigarrow \dots$ ← = weight cycling

Ex: • $f=2$ and $\bar\rho$ reducible split: $(r_0, r_1) \circlearrowleft_\delta$ $\quad (p-2-r_0, r_1+1) \overset{\delta}{\rightleftarrows} (r_0+1, p-2-r_1)$

(I don't write the det*) $\quad (p-3-r_0, p-3-r_1) \circlearrowleft_\delta$

• $f=2$ and $\bar\rho$ reducible non split

$|W(\bar\rho)| = 2 \quad W(\bar\rho) = \{(r_0, r_1), (p-2-r_0, r_1+1)\} \quad (p-2-r_0, r_1+1) \xrightarrow{\delta} (r_0, r_1) \circlearrowleft_\delta$.

(other case similar)

$\bar\rho$ semi-simple $\Rightarrow$ the map δ has nice properties, for instance there is $n \geq 1$ s.t. $\delta^n(\sigma) = \sigma$ $\Rightarrow$ δ gives a <u>partition</u> of $W(\bar\rho)$ (=cycles)

<u>Question</u>: can one "see" such cycles on $\bar\rho$?

<u>Answer</u>: Yes! : on the tensor induction $\mathrm{ind}_K^{\otimes \mathbb{Q}_p} \bar\rho := \bigotimes_{\tau \in \mathrm{Gal}(K/\mathbb{Q}_p)} \bar\rho(\tau \cdot \tau^{-1})$

Write $\left(\mathrm{ind}_K^{\otimes \mathbb{Q}_p} \bar\rho\right)\Big|_{I_{\mathbb{Q}_p}} = \bigoplus \eta$, η = fund. characters of Serre

then the <u>δ-cycles</u> have same size as the <u>Frobenius cycles</u> (at least generically)

$\eta \rightsquigarrow \eta^p \rightsquigarrow \eta^{p^2}$.

Ex: ↑ skip

$f=2$, $\bar\rho$ split $\Rightarrow (\mathrm{ind}^{\otimes} \bar\rho)\Big|_{I_{\mathbb{Q}_p}} \simeq \begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \omega_2^{p(r_0+1)+r_1+1} & 0 \\ 0 & 1 \end{pmatrix}$

$$= \omega_2^{(1+p)(r_0+1+p(r_1+1))} \circlearrowleft_{\mathrm{Frob}} \oplus \left(\omega_2^{r_0+1+p(r_1+1)} \rightleftarrows \oplus\, \omega_2^{p(r_0+1)+r_1+1}\right) \oplus 1 \circlearrowleft.$$

BP: this suggested that • $\sigma \rightsquigarrow \delta(\sigma)$ is the right weight cycling on $\pi_v[F]$

• $\sigma \in W(\bar\rho)$ should only appear on $\mathrm{soc}_{GL_2(\mathcal{O}_K)}(\pi_v[F]^{K_1})$, not on $\dfrac{\pi_v[F]^{K_1}}{\mathrm{soc}(\pi_v[F]^{K_1})}$

[as one stops as soon as one meets a weight in $W(\bar\rho)$, see Lemma 1]

<u>Prop. 2</u> (BP): ① $\exists$ a unique finite dim^l represent$^{\circ}$ $D_0(\bar\rho)$ of $GL_2(\mathbb{F}_q)$

over $\mathbb{F}$ such that: [Moreover $D_0(\bar\rho) = \bigoplus_{\sigma} D_{0,\sigma}(\bar\rho)$ → soc $= \sigma$]

(i) $\mathrm{soc}\, D_0(\bar\rho) = \bigoplus_{\sigma \in W(\bar\rho)} \sigma$ (ii) $\sigma \in W(\bar\rho)$ appears with multip. 1 in $D_0(\bar\rho)$

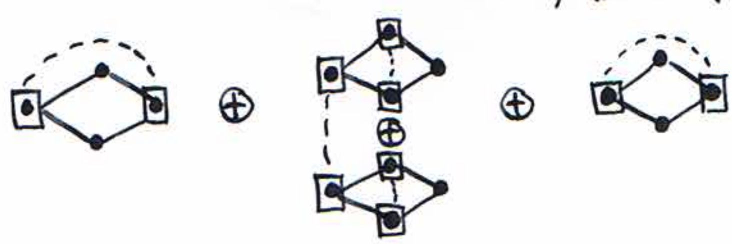
(iii) $D_0(\bar\rho)$ is maximal with respect to (i) + (ii)

② $D_0(\bar\rho)$ (as a repr. of $GL_2(\mathbb{F}_q)$) is multiplicity free; $D_0(\bar\rho)^{I_1}$ (as a repr. of I/I_1) is multiplicity free and is stable under $\chi \longleftrightarrow \chi^s$.

<u>Rk</u>: ① holds for any finite set of distinct Serre weight (not nec. $W(\bar\rho)$)

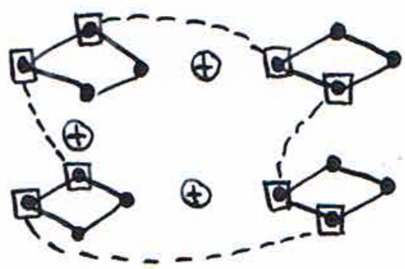
<u>Ex</u>: <u>form of $D_0(\bar\rho)$ for $f=2$</u> (• := Serre weight, — := non split ext., $\boxed{\bullet}$:= I_1-invariant, ⋯ := $\chi \leftrightarrow \chi^s$

$\bar\rho$ split:



$\bar\rho$ irreducible:

↑ skip



<u>Thm 3 (BP)</u>: ① If $\bar\rho$ is irreducible or reducible non split, one cannot write the $GL_2(\mathbb{F}_q)$-representation $D_0(\bar\rho)$ as $D \oplus D'$ where $\chi \longleftrightarrow \chi^s$ preserves D^{I_1} and D'^{I_1}.

② If $\bar\rho$ is reducible split, one has:

$$D_0(\bar\rho) = D_{0,0}(\bar\rho) \oplus D_{0,1}(\bar\rho) \oplus \cdots \oplus D_{0,f}(\bar\rho)$$

where $D_{0,i}(\bar\rho) = \bigoplus_{\lg(\sigma)=i} D_{0,\sigma}(\bar\rho)$ and each $D_{0,i}(\bar\rho)^{I_1}$ is stable under $\chi \longleftrightarrow \chi^s$.

$\lg(\sigma)$ = length of σ (see on p. ⑤)

[and is "indécomposable in the sense of ①]

(case $d>1$)

Several years after BP came:

Theorem B (Emerton, Gee, Savitt, Hu, Wang, Schraen, Le, Morra, BHHMS)

(need to increase genericity of $\bar{\rho}$)

Same assumptions as for thm. A on Serre weights + p inert in $F|F^+$ (for simplicity) + some (small) technical further assumptions on $\bar{r}$ and U^v, then there is an integer $d\geq 1$ such that:

$$\pi_v[\bar{r}]^{K_1} \simeq D_0(\bar{\rho})^{\oplus d}.$$

Rk: If $d=1$ thm. automatically implies weight cycling = previous one

③ Diagrams

Combining the actions of $GL_2(O_K)$ on $\pi_v[\bar{r}]^{K_1}$ and of $\begin{pmatrix}0&1\\p&0\end{pmatrix}$ on π^{I_1} naturally leads to the definition of diagrams.

Definition (Schneider-Stuhler, Paškūnas) A diagram is a triple (D_0, D_1, r) where D_0 = smooth repr. of $GL_2(O_K)K^\times$ over $\mathbb{F}$

D_1 = smooth repr. of $I\begin{pmatrix}0&1\\p&0\end{pmatrix}^{\mathbb{Z}}$ over $\mathbb{F}$

$r: D_1 \to D_0$ = $\underbrace{IK^\times}$-eq ivariant morphism. $(= I p^{\mathbb{Z}})$

we only consider diagrams where:

- $K^\times$ acts by a character
- $D_0 = D_0^{K_1}$ and is finite dim.
- $D_1|_{IK^\times} \simeq D_0^{I_1}$ and r is the canonical injection $D_0^{I_1} \hookrightarrow D_0^{K_1}$.

notation: $(D_1 \hookrightarrow D_0)$ (for such diagrams)

Main example: $(D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho}))$ where $p\in K^\times$ acts on $D_0(\bar{\rho})$ by $(\det(\bar{\rho})\,\mathrm{cyclo}^{-1})(p)$ and $D_1(\bar{\rho}) := D_0(\bar{\rho})^{I_1}$ with any action of $\begin{pmatrix}0&1\\p&0\end{pmatrix}$ (possible as stable under $\chi \leftrightarrow \chi^s$).

(choice of)

I now mention the following theorem, though we won't use it.

Thm. 4 (Paškūnas, BP) Assume $p > 2$ and let $(D_1 \hookrightarrow D_0)$ be a diagram as above. Then there exists a smooth admissible repr. π of $GL_2(K)$ over $\mathbb{F}$ such that: $(D_1 \hookrightarrow D_0) \hookrightarrow (\pi^{I_1} \hookrightarrow \pi^{K_1})$, $\mathrm{soc}_{GL_2(\mathcal{O}_K)} \pi = \mathrm{soc}_{GL_2(\mathcal{O}_K)} D_0$, $\pi = \langle GL_2(K) \cdot D_0 \rangle$.

[proof extensively uses injective envelopes]

It is NOT true that one can always find π as in Thm. 4 such that $\pi^{K_1} = D_0$. It is true for $D_0(\bar{\rho})$ (at least for certain actions of $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$), but the proof of this is global.

$\longrightarrow$ usual counter-example: $D_0 = \sigma \oplus \sigma^{[s]}$ (even for $GL_2(\mathbb{Q}_p)$)

Thm. 5 (BP) Assume $\bar{\rho}$ is reducible split and $(D_1 \hookrightarrow D_0) := (D_{1,\ell}(\bar{\rho}) \hookrightarrow D_{0,\ell}(\bar{\rho}))$ for $\ell \in \{0, \dots, f\}$ or $\bar{\rho}$ is irreducible and $(D_1 \hookrightarrow D_0) := (D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho}))$. Then any π as in Thm. 4 is (absolut.) irreducible.

Rk: Even if $(\pi^{I_1} \hookrightarrow \pi^{K_1}) \simeq \bigoplus_{\ell} (D_{1,\ell}(\bar{\rho}) \hookrightarrow D_{0,\ell}(\bar{\rho}))$, it doesn't imply that π is semi-simple.

Proof = explicit computations; Thm. 5 will be used!

"Parameters" in diagrams $(D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho}))$.

I first need a lemma.

Lemma (BP) Let $\chi : I \longrightarrow \mathbb{F}^\times$ such that $D_0(\bar{\rho})^{I_1}[\chi] \neq 0$, then there is a unique $i(\chi) \in \{0, \dots, q-1\}$ and a unique char. $R\chi : I \longrightarrow \mathbb{F}^\times$ such that we have an isom. of 1-dim^{l} $\mathbb{F}$-v.s.:

$$R : D_0(\bar{\rho})^{I_1}[\chi] \xrightarrow{\sim} (\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}[R\chi]$$
$$v \longmapsto \sum_{\lambda \in \mathbb{F}} \lambda^{i(\chi)} \begin{pmatrix} [\lambda] & 1 \\ 1 & 0 \end{pmatrix} v.$$

[If $\chi \in (\mathrm{soc}(D_0(\bar{\rho})))^{I_1}$, then $R\chi = \chi$]

<u>Proof</u>: Frob. reciprocity $\mathrm{Ind}_I^{GL_2(O_K)} \chi \twoheadrightarrow \langle GL_2(O_K) \cdot D_0(\bar{\rho})^I[\chi] \rangle$
+ internal structure of $\circlearrowleft$. $\square$

Let $\chi_0, \dots, \chi_{k-1}$ (for some $k \geqslant 1$) arbitrary characters of I on $D_0(\bar{\rho})^{I_1}$ such that $R(\chi_i^s) = R\chi_{i+1}$ $\forall i$ (with $\chi_k := \chi_0$).
The composition of iso. of 1-dim! v.s. :

$$(\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}[R\chi_0] \xrightarrow[\sim]{R^{-1}} D_0(\bar{\rho})^{I_1}[\chi_0] \xrightarrow[\sim]{\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)} D_0(\bar{\rho})^{I_1}[\chi_0^s] \xrightarrow[\sim]{R} (\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}[R\chi_0^s]$$

$$(\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}[R\chi_{k-1}^s] \xleftarrow[R]{\sim} \cdots \qquad \cdots \xleftarrow[R^{-1}]{\sim} (\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}[R\chi_1]$$

is a scalar $\in \mathbb{F}^\times$. <u>These scalars determine $(D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho}))$ up to isom.</u>
[for all choices of χ_i as above]

<u>Thm. C</u> (Dotto, Le, BHHMS) (case $d>1$)

Same assumptions as for Thm. B, then there is a diagram $D(\bar{\rho}) = (D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho}))$ only depending on $\bar{\rho}$ such that $(\pi_v[\mathfrak{m}]^{I_1} \hookrightarrow \pi_v[\mathfrak{m}]^{K_1}) \simeq D(\bar{\rho})^{\oplus d}$.

<u>Proof ($d=1$)</u>: prove that all above scalars only depend on $\bar{\rho}$. $\square$

[If time]
<u>Important special case (weight cycling scalars)</u>: $\chi_i \in (\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}$
i.e. $R\chi_i = \chi_i$ ($\forall i$), i.e. $\chi_i = \delta^i(\sigma_0)^{I_1}$ if $\chi_0 = \sigma_0^{I_1}$.
Let $S : (\mathrm{soc}\, D_0(\bar{\rho}))^{I_1} \xrightarrow{\sim} (\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}$ which is defined by

$$(\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}[\chi] \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)} D_0(\bar{\rho})^{I_1}[\chi^s] \xrightarrow{R} (\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}[R\chi^s].$$

Write $i(\chi^s) = \sum_{j=0}^{f-1} i(\chi^s)_j\, p^j \in \{0, \dots, q-1\}$ (previous lemma)
and $(\mathrm{soc}\, D_0(\bar{\rho}))^{I_1} = \bigoplus \mathbb{F} e_\chi$ ⟶ eigenvector for $\chi \in (\mathrm{soc}\, D_0(\bar{\rho}))^{I_1}$.

Let $(e_\chi^\vee)_\chi :=$ dual basis of (e_χ) and consider the (φ,Γ)-module (see lecture 2):

$$M(D(\bar\rho)) = \bigoplus_\chi \mathbb{F}((X))\, e_\chi^\vee$$

with $\varphi : M(D(\bar\rho)) \to M(D(\bar\rho))$ defined by:

$$\varphi(e_\chi^\vee) := \left(\prod_{j=0}^{f-1} i(\chi^s)_j!\right) X^{\sum_{j=0}^{f-1} p-1-i(\chi^s)_j} \cdot (e_\chi^\vee \circ S^{-1})$$

(one can define also a canonical action of Γ commuting with φ)

Using the values for the weight cycling parameters in Thm. C, one has:

Thm. D (B. + Dotto-Le) Same assumptions as for Thm. C, then $M(D(\bar\rho))$ is the (φ,Γ)-module of $\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\bar\rho \otimes \det(\bar\rho)^{-1})$.

(φ,Γ)-modules will be the main topic of the next lecture.
