Essen (video)
30/4/2021

Lecture 10
Conjectures for GL_n

Recall from previous lectures: $K = F_v = \mathbb{Q}_{p^f}$,

$\bar{r} : \mathrm{Gal}(\bar{F}/F) \to GL_2(\mathbb{F})$ abs. irreducible + automorphic

$\rightsquigarrow \pi_v[\bar{r}] \circlearrowleft GL_2(K)$ such that (under several technical assumptions + "minimal" case): that I will ignore

① $\underbrace{D_\xi^\vee}_{\text{defined previously}}(\pi_v[\bar{r}]) \simeq (\varphi,\Gamma)\left(\mathrm{ind}_K^{\otimes \mathbb{Q}_p} \bar{\rho}\right)$ (up to twist by some $\underbrace{\omega}_{\text{mod } p \text{ cyclo}}^*$)

(no need for minimal case here)

where $\bar{\rho} := \bar{r}|_{\mathrm{Gal}(\bar{F}_v/F_v)}$ is semi-simple + sufficiently generic

② • if $\bar{\rho}$ is irred. then $\pi_v[\bar{r}]$ is irreducible supersingular

• if $\bar{\rho}$ is red. and $f=2$, then:

$\bar{\rho}$ semi-simple $\Rightarrow \pi_v[\bar{r}] \simeq PS_1 \oplus SS \oplus PS_2$ (not known if the same)

$\bar{\rho}$ non s.s. $\Rightarrow \pi_v[\bar{r}] \simeq \underbrace{PS_1}_{\text{socle}} — SS — PS_2$ → non split ext.

• if $\bar{\rho}$ is red. and f arbitrary, one expects:

$\bar{\rho}$ s.s. $\Rightarrow \pi_v[\bar{r}] \overset{?}{\simeq} PS_1 \oplus (SS_1 \oplus \cdots \oplus SS_{f-1}) \oplus PS_2$

$\bar{\rho}$ non s.s. $\Rightarrow \pi_v[\bar{r}] \overset{?}{\simeq} PS_1 — \underbrace{SS_1 — \cdots — SS_{f-1}}_{\text{distinct SS repres.}} — PS_2$

Aim of the lecture: Give conjectural statements generalizing ① and ② to $GL_n(K)$, $n \geqslant 2$.

① $D_\xi^\vee(\pi_v[\bar{r}])$ ② "Internal structure" of $\pi_v[\bar{r}]$ ③ A few examples (↳ mainly GL_3)

① $D_\xi^\vee(\pi_v[\bar{r}])$ I recall, and generalize to $n>2$, $\pi_v[\bar{r}]$ (only deal with cpct unitary case)

F^+ = tot. real number field, F = tot. imag. quadratic ext. of F^+ ②
such that any $v|p$ in F^+ splits in F, $G/_{F^+}$ = unitary group which is GL_n at places above p and $U_n(\mathbb{R})$ at infinite places.
[As in my first talk, I ignore that there are 2 places of F above $v|p$.]
I fix $v|p$, $K = F^+_v = F_v$ <u>not necess. unramified</u> (for the moment)
$\bar{r} : \mathrm{Gal}(\bar{F}/F) \to GL_n(\mathbb{F})$ abs. irred. and U^v := c.o. s/gp of $G(\mathbb{A}^{\infty,v}_{F^+})$ such that:

$$\pi_v[\bar{r}] := \left(\varinjlim_{U_v} \left\{ f : G(F^+)\backslash G(\mathbb{A}^\infty_{F^+})/U^vU_v \longrightarrow \mathbb{F} \right\}\right)[\mathfrak{m}_{\bar{r}}] \neq 0$$

smooth admissible $\to \circlearrowleft GL_n(K) = G(F^+_v)$

<u>Conjecture:</u> | $D^\vee_\xi(\pi_v[\bar{r}])$ is finite dimensional (over $\mathbb{F}((X))$) and is the (φ,Γ)-module of $\left(\mathrm{ind}_K^{\otimes \mathbb{Q}_p}(\bar{\rho} \otimes_{\mathbb{F}} \Lambda^2\bar{\rho} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \Lambda^{n-1}\bar{\rho})\right)^{\oplus d}$ for some integer $d \geq 1$.

no assumption on K and $\bar{\rho} := \bar{r}|_{\mathrm{Gal}(\bar{K}/K)}$

<u>Rk:</u> | Can be extended to a more general global setting (i.e. not necess. compact unitary groups).

② "Internal structure" of $\pi_v[\bar{r}]$

<u>Assumptions from now on:</u> $K = F_v = \mathbb{Q}_{p^f}$ and $\bar{\rho}$ generic as follows:
- all ired cstts of $\bar{\rho}$ are distinct
- the ratio of 2 1-dim! cstts $\notin \{\omega, \omega^{-1}\}$.

<u>Ex:</u> | $n=2$, $\bar{\rho} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \Rightarrow \chi_1\chi_2^{-1} \notin \{1, \omega, \omega^{-1}\}$.

- <u>First, I go back to $n=2$, $\bar{\rho} = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$:</u> we should have (with $d=1$)

$\pi_v[\bar{r}] \overset{?}{\simeq} PS_1 — SS_1 — \cdots — SS_{f-1} — PS_2$ with:

$$V_{GL_2}(\pi_v[\bar r]) \overset{?}{\simeq} \prod_{\sigma\in Gal(K/\mathbb{Q}_p)} \chi_1^\sigma — \bigoplus_\sigma \chi_2^\sigma\Big(\prod_{\sigma'\neq\sigma}\chi_1^{\sigma'}\Big) — \cdots — \bigoplus_{\substack{I \text{ s.t.}\\ |I|=f-i}} \Big(\prod_{\sigma\in I}\chi_1^\sigma\Big)\Big(\prod_{\sigma\notin I}\chi_2^\sigma\Big) — \cdots — \prod_\sigma \chi_2^\sigma$$

see Florian's first talk

$I \subseteq Gal(K/\mathbb{Q}_p)$ $\qquad i\in\{0,\dots,f\}$

[actually implied by previous lectures]

$$V_{GL_2}(SS_i) \overset{?}{\simeq} \bigoplus_{|I|=f-i} (\quad) \quad i\in\{1,\dots,f-1\}.$$

One can rephrase this as follows: let Std := standard (algebraic) represent$^{\circ}$ of $GL_n/\mathbb{F}$ of dim. n; for $i\in\{0,\dots,f\}$ $(+n=2)$, let

$$C_i \subseteq \big(\underbrace{Std\otimes_{\mathbb{F}}\cdots\otimes_{\mathbb{F}} Std}_{Gal(K/\mathbb{Q}_p)}\big)\big|_T$$ be the <u>isotypic component</u>

of $\lambda_i := (f-i, i)\in X(T) := Hom_{gr}(T,\mathbb{G}_m)$ where $T \hookrightarrow \overbrace{GL_2\times\cdots\times GL_2}^{Gal(K/\mathbb{Q}_p)}$ diag. embedding

$\begin{pmatrix} x_1 & \\ & x_2\end{pmatrix}\in T \mapsto x_1^{f-i}x_2^{i}$

then $B\times\cdots\times B$ acts on C_i via $B\times\cdots\times B \twoheadrightarrow T\times\cdots\times T$ and one has

$\hookrightarrow$ upper Borel $\qquad$ seen as subquotient

$$(Std\otimes\cdots\otimes Std)\big|_{B\times\cdots\times B} \simeq C_0 — C_1 — \cdots — C_{f-1} — C_f .$$

socle $\qquad$ non split extension

Then
$$\begin{cases} V_{GL_2}(\pi_v[\bar r]) \simeq (Std\otimes\cdots\otimes Std)\circ(\bar\rho^\sigma)_{\sigma\in Gal(K/\mathbb{Q}_p)} \\ V_{GL_2}(SS_i) \simeq C_i\circ(\bar\rho^\sigma)_{\sigma\in Gal(K/\mathbb{Q}_p)}, \end{cases}$$

(repres. of $G_{\mathbb{Q}_p}$) $\qquad \bar\rho(\sigma\cdot\sigma^{-1})$

same with $D_\xi^\vee(PS_i)$

<u>How to predict PS or SS from C_i ?</u>

Let $\theta := (1,0)$ = highest weight of $Std|_T$, $i\in\{0,\dots,f-1\}$, $w\in W$ = weyl grp of GL_2 s.t. $w(\lambda_i)$ dominant $/B$, then:

$$C_i\circ(\bar\rho^\sigma)_\sigma \text{ is } V_{GL_2}(SS) \Leftrightarrow f\theta - w(\lambda_i) \in \mathbb{Z}_{>0}(e_1-e_2)$$

simple root of GL_2

i.e. "$C_i \hookrightarrow SS$"

• I now extend this to $GL_n(K)$, replacing $\mathrm{Std}\otimes\cdots\otimes\mathrm{Std}$ by: ④

$$\mathbb{I}^{\otimes} := \underbrace{\left(\mathrm{Std}\otimes_{\mathbb{F}}\cdots\otimes_{\mathbb{F}}\Lambda^{n-1}\mathrm{Std}\right)\otimes_{\mathbb{F}}(\text{same})\otimes_{\mathbb{F}}\cdots\otimes_{\mathbb{F}}(\text{same})}_{\mathrm{Gal}(K/\mathbb{Q}_p)}$$

seen as a repres. of $GL_n\times\cdots\times GL_n$. [could also work with $\mathrm{Res}_{K/\mathbb{Q}_p}GL_n\ldots$]

Let $\bar{\rho}\ (:= \bar{r}|_{G_K}) : G_K \longrightarrow \tilde{P}(\mathbb{F}) \hookrightarrow P(\mathbb{F}) \hookrightarrow GL_n(\mathbb{F})$

where $\begin{cases} P := \text{standard parabolic subgrp. of } GL_n \\ \tilde{P} := \text{Zariski closed algebr. subgrp of } P. \end{cases}$

containing upper Borel

Conjugating $\bar{\rho}$ by an element of $GL_n(\mathbb{F})$ $\rightsquigarrow$ can assume $\tilde{P}$, P are "as small as possible".

$\hookrightarrow$ not unique, but choose one

<u>General idea:</u> irred. cstts of $\pi_v[\bar{r}]$ $\longleftrightarrow$ isotypic components of $\mathbb{I}^{\otimes}|_{Z_{M_P}}$ via $Z_{M_P} \underset{\text{diag.}}{\hookrightarrow} GL_n\times\cdots\times GL_n$

where $\begin{cases} M_P := \text{Levi of } P \text{ containing } T \\ Z_{M_P} := \text{center of } M_P. \end{cases}$

[Ex: $\bar{\rho}$ upper triang. $\Rightarrow M_P = Z_{M_P} = T$]

Let $W(P) :=$ Weyl group of M_P, one has an injection of $\mathbb{Z}$-modules

$$X(Z_{M_P}) \hookrightarrow X(T)^{W(P)}\otimes_{\mathbb{Z}}\mathbb{Q}$$
$$\lambda \longmapsto \frac{1}{|W(P)|}\sum_{w\in W(P)} w(\lambda) =: \lambda'$$

(so, if $P = B$, $\lambda' = \lambda$).

Let $\theta := (n-1, n-2, \ldots, 0)$ = highest weight of $\mathrm{Std}\otimes_{\mathbb{F}}\Lambda^2\mathrm{Std}\otimes_{\mathbb{F}}\cdots\otimes_{\mathbb{F}}\Lambda^{n-1}\mathrm{Std}$

$\lambda\in X(Z_{M_P})$ appearing in $\mathbb{I}^{\otimes}|_{Z_{M_P}}$ and $w\in W$ s.t. $w(\lambda')$ domin

Then $f\theta - w(\lambda') \in \sum_{\alpha \in S} \mathbb{Z}_{\geq 0}\alpha$, $S :=$ simple roots of GL_n.

To $C(\lambda) :=$ isotypic component of λ in $\overline{L}^{\otimes}|_{Z_{M_P}}$ I associate

$P(C(\lambda)) :=$ standard parab. s/gp of GL_n with set of simple roots

$:=$ "support" in S of $f\theta - w(\lambda')$. $\longrightarrow$ written as above

Ex: • $n=2$, $P=B$ $\rightsquigarrow$ $P(C(\lambda_i)) = B$ if $i \in \{0, f\}$
$\lambda_i = (f-i, i)$ $\quad$ $P(C(\lambda_i)) = GL_2$ if $i \in \{1, \dots, f-1\}$

• $\lambda = f\theta|_{Z_{M_P}}$, one can check $P(C(\lambda)) = P$; in particular if $P = GL_n$ ($\Leftrightarrow \overline{\rho}$ irred), $P(C(\lambda)) = GL_n$.

One more definition is needed.

Def: A subquotient of $\overline{L}^{\otimes}|_{\tilde{P}\times\cdots\times\tilde{P}}$ is <u>good</u> if its restriction to $Z_{M_P} \hookrightarrow \tilde{P}\times\cdots\times\tilde{P}$ is a sum of isotypic components of $\overline{L}^{\otimes}|_{Z_{M_P}}$.

Any isotypic component is a good subquotient as it automatically carries an action of $\tilde{P}\times\cdots\times\tilde{P}$ via $\tilde{P}\times\cdots\times\tilde{P} \twoheadrightarrow M_P\times\cdots\times M_P$

Conjecture: (1) $\exists\, d \geq 1$ such that $\pi_v[\overline{r}] \simeq \pi^{\oplus d}$ for a finite length repres. π of $GL_n(K)$ over $\overline{\mathbb{F}}$ with distinct irr. constituents.

(2) $\exists$ a bijection ϕ between the (finite) set of subqts π' of π and the set of good subquotients of $\overline{L}^{\otimes}|_{\tilde{P}\times\cdots\times\tilde{P}}$ satisfying the following properties:

• ϕ respects inclusions and quotient maps

(⇒ ϕ induces a bijection between irr. cstts of π and isotypic components of $\bar{L}^{\otimes}|_{Z_{M_P}}$)

- if C is an isotypic component, then $\phi^{-1}(C) \simeq \mathrm{Ind}_{P(C)^-(K)}^{GL_n(K)} \pi(C)$ opposite parabolic
where $\pi(C)$ = irreducible supersingular repr. of $M_{P(C)}(K)$ over $\mathbb{F}$
- $V_{GL_n}(\pi') \simeq \phi(\pi') \circ (\bar{\rho}^{\sigma})_{\sigma \in \mathrm{Gal}(K/\mathbb{Q}_p)}$.
$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

Some comments:

only one isot. cpnt in $\bar{L}^{\otimes}|_{Z_{GL_n}}$ + $P(C) = GL_n$ (ex.)

- $\bar{\rho}$ irreducible $\Leftrightarrow \tilde{P} = P = GL_n \overset{?}{\Leftrightarrow} \pi$ is irreducible supersingular
- $\bar{\rho}$ is semi-simple $\Leftrightarrow \tilde{P} = M_P \hookrightarrow P \overset{?}{\Leftrightarrow} \pi$ is semi-simple
($\bar{L}^{\otimes}|_{M_P \times \cdots \times M_P}$ is the direct sum of its isotypic components for Z_{M_P})
- Conjecture ⇒ $\begin{matrix} D_{\xi}^{\vee} \\ V_{GL_n} \end{matrix}\Big|$ behave like exact functors on the subquotients of π
- Conjecture can be made more precise, for instance some $\pi(C)$ should be related to one another (internal symmetries)

③ A few examples

I give 2 examples for $GL_3(\mathbb{Q}_p)$, 1 for $GL_3(\mathbb{Q}_{p^2})$, all for $\bar{\rho}$ reducib

- $GL_3(\mathbb{Q}_p)$, $\bar{\rho} \simeq \begin{pmatrix} \bar{\rho}_1 & * \\ 0 & \chi_2 \end{pmatrix}$, $\begin{cases} \dim_{\mathbb{F}} \bar{\rho}_1 = 2 \\ \dim_{\mathbb{F}} \chi_2 = 1 \end{cases} \rightsquigarrow \tilde{P} = P = \boxed{\begin{smallmatrix} GL_2 & * \\ & GL_1 \end{smallmatrix}} \hookrightarrow GL_3$

One checks: $\bar{L}^{\otimes}|_{\tilde{P}} \simeq C\left((2,1,0)|_{Z_{M_P}}\right) — C\left((1,1,1)|_{Z_{M_P}}\right) — C\left((0,1,2)|_{Z_{M_P}}\right)$

see Example $\rightsquigarrow P(C) = P$ see below see below

$\lambda = (1,1,1) \Rightarrow \lambda' = \lambda = (1,1,1) \rightsquigarrow \theta - \lambda' = (2,1,0) - (1,1,1) = (1,0,-1) = (e_1 - e_2) + (e_2 - e_3)$

⇒ corresponding $P(C)$ is GL_3

$\lambda = (0,1,2) \Rightarrow \lambda' = \left(\frac{1}{2}, \frac{1}{2}, 2\right) \Rightarrow w(\lambda') = \left(2, \frac{1}{2}, \frac{1}{2}\right)$ (dominant)

$\Rightarrow \theta - w(\lambda') = (2,1,0) - (2, 1/2, 1/2) = 1/2(e_2 - e_3) \Rightarrow$ corresponding

$P(c)$ is $\begin{smallmatrix} GL_1 & \\ & GL_2 \end{smallmatrix}$, so one should have:

$$\pi \simeq \left(\mathrm{Ind}_{\begin{smallmatrix} GL_2 & \\ & GL_1 \end{smallmatrix}}^{GL_3} ss_1\right) — SS — \left(\mathrm{Ind}_{\begin{smallmatrix} GL_1 & \\ & GL_2 \end{smallmatrix}}^{GL_3} ss_2\right)$$

with $V_{GL_3}(\mathrm{Ind}\, ss_1) \simeq \bar{\rho}_1 \otimes \det(\bar{\rho}_1)$, $V_{GL_3}(\mathrm{Ind}) = \bar{\rho}_1^{\otimes 2} \otimes \chi_2 \oplus \det(\bar{\rho}_1)\chi_2$

$V_{GL_3}(\mathrm{Ind}\, ss_2) \simeq \bar{\rho}_1 \otimes \chi_2^2$. (can be more specific about ss_1, ss_2)

$\bar{\rho} \simeq \begin{pmatrix} \bar{\rho}_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \Rightarrow$ semi-simplify everything.

- $GL_3(\mathbb{Q}_p)$, $\bar{\rho} = \begin{pmatrix} \chi_1 & & * \\ & \chi_2 & \\ 0 & & \chi_3 \end{pmatrix}$, $\dim \chi_i = 1 \leadsto \tilde{P} = P = B$ (upper Borel)

then one finds from $\bar{L}^{\otimes}|_B$:

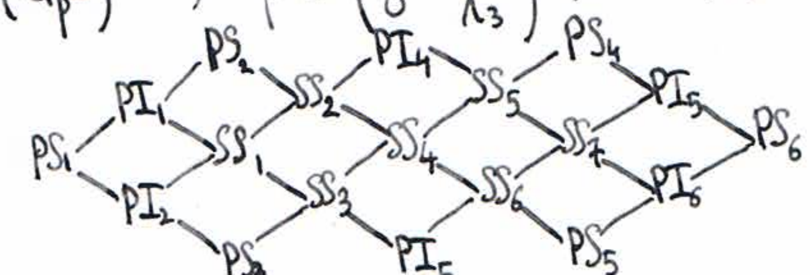
$$\pi \simeq \begin{tikzcd}[column sep=small, row sep=small] & PS_2 \arrow[dr] & & PS_4 \arrow[dr] & \\ PS_1 \arrow[ur] \arrow[dr] & & SS \arrow[ur] \arrow[dr] & & PS_6 \\ & PS_3 \arrow[ur] & & PS_5 \arrow[ur] & \end{tikzcd} \xmapsto{V_{GL_3}} \begin{tikzcd}[column sep=tiny, row sep=small] & \chi_1\chi_2^2 \arrow[dr] & & \chi_1\chi_3^2 \arrow[dr] & \\ \chi_1^2\chi_2 \arrow[ur] \arrow[dr] & & (\chi_1\chi_2\chi_3)^{\oplus 3} \arrow[ur] \arrow[dr] & & \chi_2\chi_3^2 \\ & \chi_1^2\chi_3 \arrow[ur] & & \chi_2^2\chi_3 \arrow[ur] & \end{tikzcd}$$

$\bar{\rho} = \begin{pmatrix} \chi_1 & & 0 \\ & \chi_2 & \\ 0 & & \chi_3 \end{pmatrix} \Rightarrow$ semi-simplify.

$\bar{\rho} = \begin{pmatrix} \chi_1 & * & * \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix} \leadsto \tilde{P} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \leadsto P = B$, one finds from $\bar{L}^{\otimes}|_{\tilde{P}}$:

$$\begin{tikzcd}[column sep=small, row sep=small] & PS_2 \arrow[dr] & \\ PS_1 \arrow[ur] \arrow[dr] & & PS_5 \\ & SS \arrow[ur] \arrow[dl] \arrow[dr] & \\ PS_3 \arrow[dr] & & PS_6 \\ & PS_4 \arrow[ur] & \end{tikzcd} \xmapsto{V_{GL_3}} \begin{matrix} \chi_1^2\chi_2 < \\ \\ \chi_1^2\chi_3 < \end{matrix} \quad \cdots$$

- $GL_3(\mathbb{Q}_{p^2})$, $\bar{\rho} = \begin{pmatrix} \chi_1 & & * \\ & \chi_2 & \\ 0 & & \chi_3 \end{pmatrix}$, $\dim \chi_i = 1 \leadsto \tilde{P} = P = B$ and

$\pi \simeq$ 

$\begin{cases} 6\ PS \leadsto V_{GL_3} \dim 1 \\ 6\ PI \leadsto V_{GL_3} \dim 2 \\ 6\ SS \leadsto V_{GL_3} \dim 8 \\ 1\ SS \leadsto V_{GL_3} \dim 15 \end{cases}$

I would like to finish with a "numerical coincidence" (still in that case of $GL_3(\mathbb{Q}_{p^2})$). $\rightarrow$ (and which also works for $f=1$!)

Assume $\bar{\rho} = \bar{\rho}^{ss} = \chi_1 \oplus \chi_2 \oplus \chi_3 \Rightarrow$ 81 Serre weights of $GL_3(\mathbb{F}_{p^2})$ associated to $\bar{\rho}$ (Herzig); by work of Le Hung, Le, Levin, Morra, they all occur in $\mathrm{soc}_{GL_3(\mathbb{Z}_{p^2})}(\pi_v[\bar{r}])$. $\rightarrow$ take the "first Sw. you meet"

For $\bar{\rho}$ sufficiently generic, there is a natural candidate for the weight cycling $\left(\text{under both } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ p & 0 & 0 \\ 0 & p & 0 \end{pmatrix}\right)$ which gives a partition of the 81 Serre weights of the following form:

$$\begin{cases} 6 \text{ packets of } 1 \text{ S.w.} \\ 6 \text{ packets of } 2 \text{ S.w.} \\ 6 \text{ packets of } 8 \text{ S.w.} \\ 1 \text{ packet of } 15 \text{ S.w.} \end{cases}$$

So this very much suggests:

$$\begin{cases} \mathrm{soc}_{GL_3(\mathbb{Z}_{p^2})}(PS) = 1 \text{ S.w.} \quad (OK) \\ \mathrm{soc}_{GL_3(\mathbb{Z}_{p^2})}(PI) = 2 \text{ S.w.} \\ \mathrm{soc}_{GL_3(\mathbb{Z}_{p^2})}(SS) = 8 \text{ S.w. for } 6 \text{ SS} \\ \mathrm{soc}_{GL_3(\mathbb{Z}_{p^2})}(SS = 15 \text{ S.w. for the last SS} \end{cases}$$