Lecture 6
Self-duality, finiteness results in semisimple case

Yongquan Hu

Morningside Center of Mathematics

Essen Spring School
April 28, 2021

Notation. Keep (mostly) the notation in previous lectures.

- K = unramified extension over $\mathbb{Q}_p$ of degree f ;
- $\mathcal{O}_K$ = integers of K, $\mathbb{F}_q \cong \mathcal{O}_K/p$;
- $G = \mathrm{GL}_2(K)$, Z =center ;
- $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $\overline{B} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$;
- I = Iwahori, I_1 = pro-p-Iwahori, $H := \begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix} \cong I/I_1$;
- $K_1 = \mathrm{Ker}\big(\mathrm{GL}_2(\mathcal{O}_K) \to \mathrm{GL}_2(\mathbb{F}_q)\big)$, $Z_1 = Z \cap K_1$;
- $(E, \mathcal{O}, \mathbb{F})$ = rings of coefficients.

Fix $\bar{\rho} : G_K := \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ cont., written in usual form with genericity conditions (slightly modified) :

- **generic** : if $10 \leq r_i \leq p - 12$ (as in [BHHMS1]) ;
- **strongly generic** : $\max\{10, 2f\} \leq r_i \leq p - \max\{12, 2f + 2\}$ (as in [BHHMS2]).

Let $\pi_v(\bar{r})$ = smooth admissible representation of G corresponding to some globalization $\bar{r}$ of $\bar{\rho}$ in mod p cohomology (cf. Lecture 1).

Start with :

Fact : if π is an irreducible smooth admissible $\mathbb{C}$-representation of $\mathrm{GL}_2(K)$, then

$$\hat{\pi} \cong \pi \otimes (\zeta^{-1} \circ \det),$$

where $\hat{\pi} := \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})^{\infty}$ denotes the *contragredient* (or smooth dual) of π, and $\zeta =$ the central character of π.

Galois side : 2-dimensional representation is dual to itself up to twist :

if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, then

$${}^{T}g^{-1} = \det(g)^{-1} \cdot (w^{-1} g w).$$

However, if π is over $\mathbb{F}$, $\hat{\pi}$ is usually zero (Livné, Vignéras) !

Set $\Lambda := \mathbb{F}[\![P]\!]$, where $P =$ pro-p open subgroup of G.
For finitely generated Λ-module M, set

$$\mathrm{E}^i(M) := \mathrm{Ext}^i_\Lambda(M, \Lambda).$$

Recall : Λ has global dimension $4f$.

Define

$$\text{grade} \quad j_\Lambda(M) := \min\{i \geq 0 : \mathrm{E}^i(M) \neq 0\},$$

$$\text{dimension} \quad \delta_\Lambda(M) := \mathrm{gld}(\Lambda) - j_\Lambda(M).$$

Say M is **Cohen-Macaulay** if $j_\Lambda(M) = \mathrm{pd}_\Lambda(M)$. (e.g. projective $\Rightarrow$ CM)

Remark (Venjakob) :

- $\mathrm{pd}_\Lambda(M) = \max\{i \geq 0 : \mathrm{E}^i(M) \neq 0\}$.
- Λ satisfies **Auslander condition** : for any $N \subset \mathrm{E}^j(M)$, $j_\Lambda(N) \geq j$.

Now consider $\mathfrak{C}_G :=$ category of f.g. (left) Λ-modules together with a compatible action of G. (**Example** : $\pi^\vee \in \mathfrak{C}_G$ for admissible π).
Then $\mathrm{E}^i(M) \in \mathfrak{C}_G$.

Definition

Let $M \in \mathfrak{C}_G$ be Cohen-Macaulay. We say M is *essentially self-dual*, if

$$\mathrm{E}^{j_\Lambda(M)}(M) \cong M \otimes (\zeta \circ \det)$$

for some ζ.

Example. Below $i = 3f$.

(a) (Kohlhaase) $\mathrm{E}^i((\mathrm{Ind}_B^G \chi)^\vee) \cong (\mathrm{Ind}_B^G \chi^{-1}\alpha_B)^\vee$, where $\alpha_B := \omega \otimes \omega^{-1}$. Hence

$$\mathrm{E}^i\big((\mathrm{Ind}_B^G \chi_1\omega^{-1} \otimes \chi_2)^\vee\big) \cong (\mathrm{Ind}_B^G \chi_2\omega^{-1} \otimes \chi_1)^\vee \otimes (\zeta \circ \det).$$

(b) $K = \mathbb{Q}_p$, π supersingular, then ([Koh])

$$\mathrm{E}^i(\pi^\vee) \cong \pi^\vee \otimes (\zeta \circ \det).$$

(c) Let $\widetilde{H}^0$ be the space of *mod p* modular forms of level U^v in global setting (ii) or (iii) of Lecture 1 (or $\widetilde{H}^1$ in setting (i)).

Theorem (Calegari-Emerton, Hill). $(\widetilde{H}^0_{\mathfrak{m}})^\vee$ is projective (hence Cohen-Macaulay), and have $\mathbb{T} \times G$-equivariant isomorphism

$$\mathrm{E}^0((\widetilde{H}^0_{\mathfrak{m}})^\vee) \cong (\widetilde{H}^0_{\mathfrak{m}})^\vee \otimes \zeta.$$

Theorem 1

If $\mathrm{GK}(\pi_v(\bar{r})) \leq f$, then $\pi_v(\bar{r})^\vee$ is essentially self-dual.

Recall (Lecture 1) : $\delta_\Lambda(\pi_v(\bar{r})^\vee)$ is denoted $\mathrm{GK}(\pi_v(\bar{r}))$.

Patching module

Let $R_\infty = R_{\bar{\rho}}^{\square}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[\![x_1,\ldots,x_r]\!]$. Following [CEGGPS], a patching module is a non-zero $R_\infty[G]$-module $\mathbb{M}_\infty$ satisfying (among others) :

- $\mathbb{M}_\infty/\mathfrak{m}_\infty \cong \pi_v(\bar{r})^\vee$;
- $\mathbb{M}_\infty$ is a finitely generated $R_\infty[\![\mathrm{GL}_2(\mathcal{O}_K)]\!]$-module ;
- $\exists$ regular local ring S_∞ (together with $S_\infty \to R_\infty$), such that $\mathbb{M}_\infty$ is f.g. projective $S_\infty[\![\mathrm{GL}_2(\mathcal{O}_K)]\!]$-module. Moreover,

$$\mathbb{M}_\infty \otimes_{S_\infty} \mathbb{F} \cong (\widetilde{H}^0_{\mathfrak{m}})^\vee.$$

Theorem 1 follows from Example (c) and :

Theorem (Miracle flatness, [GN])

Assume $R_{\bar{\rho}}^{\square}$ is **regular**. If

$$\mathrm{GK}(\pi_v(\bar{r})) \leq f,$$

then the equality holds, $\mathbb{M}_\infty$ is flat over R_∞, and $\mathbb{T}_{\mathfrak{m}}$ is complete intersection.

(**Caution** : It is crucial that $R_{\bar{\rho}}^{\square}$ is regular.)

Notation. Write $\Lambda = \mathbb{F}[\![I_1/Z_1]\!]$ from now on.

Recall (Lecture 5) : $\mathrm{gr}_{\mathfrak{m}_{I_1}}(\Lambda)$ is isomorphic to

$$\bigotimes_{i=0}^{f-1} \mathbb{F}[y_i, z_i, h_i]$$

where $[y_i, z_i] = h_i$, $[h_i, y_i] = [h_i, z_i] = 0$, and variables with $i \neq j$ commute. Moreover, $\deg(y_i) = \deg(z_i) = 1$.

The action of $g \in H := \begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix}$:

$$g \cdot y_i = \alpha(g)^{p^i} y_i, \quad g \cdot z_i = \alpha(g)^{-p^i} z_i, \quad g \cdot h_i = h_i$$

where α sends $\begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix}$ to ad^{-1} (via fixed embedding $\mathbb{F}_q \hookrightarrow \mathbb{F}$).

Let

$$J := (y_i z_i, z_i y_i,\ 0 \le i \le f-1).$$

Lemma (cf. Lecture 5)

$\mathrm{gr}(\Lambda)/J$ is isomorphic to the commutative ring

$$\mathbb{F}[y_i, z_i; 0 \le i \le f-1]/(y_i z_i; 0 \le i \le f-1).$$

- If N is a f.g. $\mathrm{gr}(\Lambda)$-module killed by a power of J, can define $m_{\mathfrak{p}}(N)$, where $\mathfrak{p}$ is a minimal prime ideal of $\mathrm{gr}(\Lambda)/J$. Let

$$\mathfrak{p}_0 := (z_0, \dots, z_{f-1}).$$

- Let $\mathcal{C}_1$ be the category of smooth adm. π (with central character) such that $\mathrm{gr}(\pi^\vee)$ is killed by a power of J (cf. Lecture 5). This is an abelian category and stable under extensions and $\mathrm{E}^i_\Lambda(-)$.

Main result

Write $m_{\mathfrak{p}}(\pi)$ for $m_{\mathfrak{p}}(\mathrm{gr}(\pi^\vee))$ and $\mathbb{V} = \mathbb{V}_{\mathrm{GL_2}}$.

Theorem (cf. Lecture 5)

For $\pi \in \mathcal{C}_1$, $\dim_{\mathbb{F}} \mathbb{V}(\pi) \leq m_{\mathfrak{p}_0}(\pi)$.

Theorem 2 (BHHMS)

Let $\bar{\rho}$ be semisimple and generic. Then $\pi_v(\bar{r}) \in \mathcal{C}_1$ and $m_{\mathfrak{p}_0}(\pi_v(\bar{r})) \leq 2^f r$.

Together with the lower bound in Lecture 4, we deduce

Corollary 3 (BHHMS2)

If $\bar{\rho}$ is semisimple and strongly generic, then $\mathbb{V}(\pi_v(\bar{r})) \cong (\mathrm{ind}_K^{\otimes \mathbb{Q}_p} \bar{\rho})^{\oplus r}$.

Assume $r = 1$ (for simplicity).

Key ingredient : "multiplicity free property"

$$[\pi_v(\overline{r})[\mathfrak{m}_{I_1}^3] : \chi] = [\pi_v(\overline{r})^{I_1} : \chi] = 1, \quad \forall \chi \in \pi_v(\overline{r})^{I_1}.$$

This will be proved in Lecture 7, 8.

Remark. This multiplicity-freeness implies immediately $\mathrm{GK}(\pi_v(\overline{r})) \leq f$.

If $\pi_v(\overline{r})[\mathfrak{m}_{I_1}^3]$ is multiplicity free, then $y_i z_i$ and $z_i y_i$ act trivially on $\mathrm{gr}^0(\pi_v(\overline{r})^\vee)$. Thus $\mathrm{gr}(\pi_v(\overline{r})^\vee)$ is finitely generated module over $\mathrm{gr}(\Lambda)/J \cong \mathbb{F}[y_i, z_i]/(y_i z_i)$, which has dimension f. □

Example. For $f = 1$, $\mathrm{gr}^{\geq -2}(F[\![I_1/Z_1]\!])$ looks like

gr^0	1
gr^{-1}	$\alpha \oplus \alpha^{-1}$
gr^{-2}	$\alpha^2 \oplus 1 \oplus 1 \oplus \alpha^{-2}$

The proof

We construct an explicit graded $\mathrm{gr}(\Lambda)$-module N with $m_{\mathfrak{p}_0}(N) = 2^f$, s.t.

$$N \twoheadrightarrow \mathrm{gr}(\pi_v(\bar{r})^\vee).$$

The proof is of combinatorial nature, and uses the fact that $\pi_v(\bar{r})[\mathfrak{m}_{I_1}^3]$ is multiplicity free (under weak genericity condition).

Remark : An obvious such module is

$$N' := \bigoplus_{\chi \in \pi_v(\bar{r})^{I_1}} (\mathrm{gr}(\Lambda)/J) \otimes \chi^\vee.$$

But $m_{\mathfrak{p}_0}(N') = \dim \pi_v(\bar{r})^{I_1}$ (as $m_{\mathfrak{p}_0}(\mathrm{gr}(\Lambda)/J) = 1$), which (often) $> 2^f = |W(\bar{\rho})|$.

Proof of Thm.2 Divide $\pi_v(\bar{r})^{I_1}$ as $\mathcal{P} \cup \mathcal{P}^c$, with

$$\mathcal{P} := \{\sigma^{I_1} : \sigma \in W(\bar{\rho})\}.$$

Then N has the form

$$N = \Big(\bigoplus_{\chi\in\mathcal{P}} (\mathrm{gr}(\Lambda)/\mathfrak{a}_\chi) \otimes \chi^\vee\Big) \oplus \Big(\bigoplus_{\chi\in\mathcal{P}^c} (\cdots)\Big)$$

for suitable $\mathfrak{a}_\chi$ (containing J) determined by relations between χ's.

Key fact : for $\chi \in \mathcal{P}^c$, $y_i \in \mathfrak{a}_\chi$ for some i.

Thus $m_{\mathfrak{p}_0}(\oplus_{\chi\in\mathcal{P}^c}(\cdots)) = 0$, and $m_{\mathfrak{p}_0}(N) \leq |\mathcal{P}| = 2^f$.

(Actually this is an equality.)

Example 1. $f = 1$.

(a) If $\bar{\rho}$ is irreducible, then $\pi_v(\bar{r})$ is supersingular, and $\dim \pi_v(\bar{r})^{I_1} = 2$ (Breuil, cf. Lecture 2), then take $N = N'$ (Thm of Paškūnas).

(b) If $\bar{\rho}$ is reducible split, then $\pi_v(\bar{r}) = \pi_0 \oplus \pi_1$ (both PS), and

$$\pi_v(\bar{r})^{I_1} = (\chi_{\sigma_0} \oplus \chi_{\sigma_0}^s) \oplus (\chi_{\sigma_1} \oplus \chi_{\sigma_1}^s).$$

Fact : $\chi_{\sigma_0}^s = \chi_{\sigma_1}\alpha$, $\chi_{\sigma_1}^s = \chi_{\sigma_0}\alpha$.

Let $\{e_0, e_0', e_1, e_1'\}$ be dual basis of $\pi_v(\bar{r})^{I_1}$. By multiplicity-freeness,

$$z \cdot e_0 = z \cdot e_1 = y \cdot e_0' = y \cdot e_1' = 0.$$

Take N to be

$$(\mathrm{gr}(\Lambda)/(z,h) \otimes \chi_{\sigma_0}^{\vee}) \oplus (\mathrm{gr}(\Lambda)/(y,h) \otimes (\chi_{\sigma_0}^s)^{\vee}) \oplus (\mathrm{others}).$$

Example 2. $f = 2$, $\bar{\rho}$ irreducible. Then $W(\bar{\rho}) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where

$$\sigma_1 = (r_0, r_1), \quad \sigma_2 = (r_0 - 1, p - 2 - r_1)$$

$$\sigma_3 = (p - 1 - r_0, p - 3 - r_1), \quad \sigma_4 = (p - 2 - r_0, r_1 + 1)$$

(up to twist, cf. Lecture 3). Moreover,

$$\pi_v(\bar{r})^{I_1} \cong \bigoplus_{i=1}^{4} (\chi_{\sigma_i} \oplus \chi^s_{\sigma_i}).$$

One checks $\chi^s_{\sigma_3} = \chi_{\sigma_1}\alpha^p$, etc.

By multiplicity freeness of $\pi_v(\bar{r})[\mathfrak{m}^3_{I_1}]$, take N to be

$$(\mathrm{gr}(\Lambda)/(J, z_1) \otimes \chi^\vee_{\sigma_1}) \oplus (\mathrm{gr}(\Lambda)/(J, y_1) \otimes (\chi^s_{\sigma_3})^\vee) \oplus (\mathrm{others}).$$

Assume $\bar{\rho}$ is semisimple and strongly generic.

Theorem 4 (BHHMS2)

As a G-representation, $\pi_v(\bar{r})$ can be generated by $D_0(\bar{\rho})$.

The proof uses the computation of (φ, Γ)-modules attached to $\pi_v(\bar{r})$ (Lecture 4).

Remark. The non-semisimple case (under weaker genericity condition) will be treated in Lecture 9 (due to HW, the proof is of different nature).

Lemma 5

Let π' a subquotient of $\pi_v(\bar{r})$.

(i) $\dim_{\mathbb{F}} \mathbb{V}(\pi') = m_{\mathfrak{p}_0}(\pi')$.

(ii) If π' is a subrepresentation of $\pi_v(\bar{r})$, then

$$\dim_{\mathbb{F}} \mathbb{V}(\pi') = m_{\mathfrak{p}_0}(\pi') = \lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi').$$

In particular, $\mathbb{V}(\pi') \neq 0$ if $\pi' \neq 0$.

(iii) If π' is a quotient of $\pi_v(\bar{r})$ and $\pi' \neq 0$, then $\mathbb{V}(\pi') \neq 0$.

Proof of Thm.4 Let $\pi_1 := \langle G.D_0(\bar{\rho})\rangle$ and $\pi_2 := \pi/\pi_1$. Equiv. to show : $\pi_2 = 0$.

From **Lemma 5**(ii) :

$$\dim \mathbb{V}(\pi_1) = \dim \mathbb{V}(\pi_v(\bar{r})).$$

Since $\mathbb{V}(-)$ is exact, we deduce $\mathbb{V}(\pi_2) = 0$, then conclude by (iii).

Proof of Lemma 5. (i) Always have inequality (recalled above) :

$$\dim_{\mathbb{F}} \mathbb{V}(-) \leq m_{\mathfrak{p}_0}(-).$$

For $\pi_v(\bar{r})$, we have equality

$$\dim_{\mathbb{F}} \mathbb{V}(\pi_v(\bar{r})) \leq m_{\mathfrak{p}_0}(\pi_v(\bar{r})).$$

The claim follows.

(ii) First, the construction of N (see **Thm. 2**) shows

$$m_{\mathfrak{p}_0}(\pi') \leq \lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi').$$

Indeed, write $N = \oplus_{\chi\in\mathcal{P}\cup\mathcal{P}^c} N_\chi$, let $\mathcal{P}' = \left\{\sigma^{I_1} : \ \sigma \in \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi')\right\}$, then

$$m_{\mathfrak{p}_0}(\pi') \leq m_{\mathfrak{p}_0}(\oplus_{\chi\in\mathcal{P}'} N_\chi).$$

Second, $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi'$ has a special form, being a direct sum of "orbits". Namely, if $\tau_\ell := \oplus_{\ell(\sigma)=\ell}\sigma$, then

$$(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi') \cap \tau_\ell = 0 \text{ or } \tau_\ell.$$

By the computation of (φ, Γ)-modules (Lecture 4), each $\langle G.\tau_\ell \rangle$ contains an *admissible* submodule which gives a (φ, Γ)-module of rank equal to $\lg(\tau_\ell)$. Thus

$$\dim_{\mathbb{F}} \mathbb{V}(\pi') \geq \lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi').$$

(iii) This uses crucially the (essential) self-duality of $\pi_v(\bar{r})$.

Let π'' be $\mathrm{Ker}(\pi_v(\bar{r}) \twoheadrightarrow \pi')$, so that

$$0 \to \pi''^\vee \to \pi_v(\bar{r})^\vee \to \pi'^\vee \to 0.$$

It induces

$$0 \to \mathrm{E}^{2f}(\pi''^\vee) \to \mathrm{E}^{2f}(\pi_v(\bar{r})^\vee) \xrightarrow{\gamma} \mathrm{E}^{2f}(\pi'^\vee) \to \mathrm{E}^{2f+1}(\pi''^\vee);$$

let $\widetilde{\pi}'$ be the dual of $\mathrm{Image}(\gamma) \otimes \zeta^{-1}$.
Since $\mathrm{E}^{2f}(\pi_v(\bar{r})^\vee) \otimes \zeta^{-1} \cong \pi_v(\bar{r})^\vee$, $\widetilde{\pi}'$ is a sub-rep. of $\pi_v(\bar{r})$.

Claim. $\widetilde{\pi}' \neq 0$.

Since $\pi_v(\bar{r})$ is CM, it is **pure** : any non-zero submodule has the same grade as $\pi_v(\bar{r})^\vee$. Thus,

$$j(\pi'^\vee) = j(\pi_v(\bar{r})^\vee) = 2f.$$

Thus $\mathrm{E}^{2f}(\pi'^\vee) \neq 0$, and $\mathrm{Im}(\gamma) \neq 0$ (by Auslander condition as $j(\mathrm{E}^{j(M)}(M)) = j$!).

Notice that π' and $\widetilde{\pi}'$ have the same "characteristic cycle", i.e. $m_{\mathfrak{p}}(\pi') = m_{\mathfrak{p}}(\widetilde{\pi}')$ for any $\mathfrak{p}$. This is because

$$m_{\mathfrak{p}}(\mathrm{E}^{2f+1}(\pi''^{\vee})) = 0$$

for any $\mathfrak{p}$ (again by the Auslander condition) and the following general fact (here twisting is not important as we only look at $m_{\mathfrak{p}}(-)$) :

$$m_{\mathfrak{p}}(\pi'^{\vee}) = m_{\mathfrak{p}}(\mathrm{E}^{2f}(\pi'^{\vee})).$$

Hence

$$\dim_{\mathbb{F}} \mathbb{V}(\pi') = m_{\mathfrak{p}_0}(\pi') = m_{\mathfrak{p}_0}(\widetilde{\pi}') = \dim_{\mathbb{F}} \mathbb{V}(\widetilde{\pi}'),$$

and left to show $\mathbb{V}(\widetilde{\pi}') \neq 0$. However, $\widetilde{\pi}'$ is a **sub-rep.** of $\pi_v(\bar{r})$, so conclude by (ii). □

1. Self-duality of $\pi_v(\bar{r})$
2. Upper bound of $\dim_{\mathbb{F}} \mathbb{V}(\pi_v(\bar{r}))$
3. Finite generation (I) : semisimple case
4. The length

We assume from now on $r = 1$ (NOT for simiplicity).

Theorem 5 (BHHMS2)

(i) If $\bar{\rho}$ is irreducible, then $\pi_v(\bar{r})$ is irreducible.

(ii) If $\bar{\rho}$ is reducible split, then

$$\pi_v(\bar{r}) \cong \pi_0 \oplus \pi' \oplus \pi_f$$

with π_0, π_f principal series. If moreover $f = 2$, then π' is (irreducible) supersingular.

Remark

(i) In general, it is not clear if "finite generated $\Longrightarrow$ finite length".

(ii) [BP, Thm. 19.10(ii)] (which says that if $\bar{\rho}$ is reducible split then π in their construction is also semisimple) does not apply here.

Proof of Thm.5 (i) Follows from Lecture 3, once we know $\pi_v(\bar{r})$ is generated by $D_0(\bar{\rho})$ by **Theorem 4**.

(ii) Since $\bar{\rho}$ is reducible split, $W(\bar{\rho})$ contains two special Serre weights σ_0, σ_f, where (up to twist)

$$\sigma_0 = (r_0, r_1, \cdots, r_{f-1}), \quad \sigma_f = (p-3-r_0, p-3-r_1, \cdots, p-3-r_{f-1}),$$

Let $\pi_0 := \langle \mathrm{GL}_2(K).\sigma_0 \rangle$.

Claim : π_0 **is a principal series**.

It is proved by a weight cycling argument (Lecture 3) together with

$$\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_{\sigma_0}) \cap W(\bar{\rho}) = \{\sigma_0\}.$$

Proof of Claim. If $0 \neq v_0 \in \sigma_0^{I_1}$, the vector $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} v_0 \in \pi^{I_1}$ with eigenchar. $\chi_{\sigma_0}^s$, and induces

$$\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_{\sigma_0}^s \to \pi_0.$$

It is injective by $(*)$.

Consider the induced morphism

$$\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)Z}^G \sigma_0 \twoheadrightarrow \pi_0,$$

let T be the Hecke operator (Barthel-Livné), then $T(\sigma_0) \neq 0$.

But σ_0 occurs in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} \pi_v(\bar{r})$ with multiplicity 1, so $T(\sigma_0) = \lambda_0 \sigma_0$ for some $\lambda_0 \in \mathbb{F}^\times$, i.e. $\pi_0 \cong \text{c-Ind}(\sigma_0)/(T - \lambda_0)$ is an irred. PS. □

Similarly, $\pi_f := \langle G.\sigma_f \rangle$ is also PS. In particular,

$$\pi_0 \oplus \pi_f \hookrightarrow \pi_v(\bar{r}).$$

The essential self-duality of $\pi_v(\bar{r})$, and of $\pi_0 \oplus \pi_2$ (Example (a)), implies that $\pi_v(\bar{r})$ also admits $\pi_0 \oplus \pi_f$ as a quotient.
Need : the composition

$$\pi_0 \oplus \pi_f \hookrightarrow \pi_v(\bar{r}) \twoheadrightarrow \pi_0 \oplus \pi_f$$

is an isomorphism.

Note that $\pi_v(\bar{r})$ can be generated by $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi_v(\bar{r}) \cong \oplus_{\sigma \in W(\bar{\rho})}\sigma$. So the composition

$$\iota : \oplus_\sigma \sigma \hookrightarrow \pi_v(\bar{r}) \twoheadrightarrow \pi_0$$

is nonzero. The image being contained in the socle of π_0, (i.e. σ_0), $\iota|_{\sigma_0}$ is nonzero, hence induces $\pi_0 \xrightarrow{\sim} \pi_0$. □

Some facts for later use :

$\mathbb{M}_\infty$ defines an exact functor : $\mathcal{O}[\mathrm{GL}_2(\mathcal{O}_K)]$-**Mod** $\longrightarrow R_\infty$-**Mod**

$$M_\infty(\Theta) := \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathbb{M}_\infty, \Theta^d)^d.$$

(a) For $\lambda = (a_j, b_j)_{0\leq j\leq f-1}$ with $a_j > b_j$, and $\tau : I_K \to \mathrm{GL}_2(E)$ an inertial type, set

$$V(\lambda-\eta) := \bigotimes_{0\leq j\leq f-1} \left((\mathrm{Sym}^{a_j-b_j-1}E^2)\otimes \det{}^{b_j}\right)^{\mathrm{Fr}^j}$$

with $\eta = (1,0)$, and $\sigma(\tau) :=$ smooth irred. rep (over E) of $\mathrm{GL}_2(\mathcal{O}_K)$ by Henniart's inertial LLC.

If $\Theta \subset V(\lambda-\eta)\otimes\sigma(\tau)$ is an $\mathcal{O}[\mathrm{GL}_2(\mathcal{O}_K)]$-lattice, then $M_\infty(\Theta)$ is maximal CM and the action of R_∞ factors through $R_\infty \otimes_{R^{\square}_{\bar{\rho}}} R^{\lambda,\tau}_{\bar{\rho}}$, where $R^{\lambda,\tau}_{\bar{\rho}}$ is Kisin's pot. semistable deformation ring of type (λ,τ).

(b) $M_\infty(\Theta)/\mathfrak{m}_\infty \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta, \pi_v(\bar{r}))^\vee$.

In particular, $M_\infty(\Theta)$ is a cyclic R_∞-module iff

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta/\varpi\Theta, \pi_v(\bar{r})) = 1.$$

Example. $[\pi_v(\bar{r})^{K_1} : \sigma] = 1$ if and only if $M_\infty(\mathrm{Proj}_{\mathrm{GL}_2(\mathbb{F}_q)}\sigma)$ is a cyclic R_∞-module.

(c) The flatness of M_∞ over R_∞ induces a Koszul type resolution of $\pi_v(\bar{r})^\vee$ in terms of M_∞ :

$$\cdots \to M_\infty^{\oplus\binom{n}{2}} \to M_\infty^{\oplus n} \to M_\infty \to \pi_v(\bar{r})^\vee \to 0.$$