Lecture 9
Finiteness results in non-semisimple case

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1. GK dimension of $\pi_v(\bar{r})$
2. Generation by $D_0(\bar{\rho})$
3. Finite length when $f = 2$

Notation. Keep (mostly) the notation in previous lectures.

- K = unramified extension over $\mathbb{Q}_p$ of degree f ;
- $\mathcal{O}_K$ = integers of K, $\mathbb{F}_q \cong \mathcal{O}_K/p$;
- $G = \mathrm{GL}_2(K)$, Z =center ;
- $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $\overline{B} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$, $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$;
- I =Iwahori, I_1 =pro-p-Iwahori, $H := \begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix} \cong I/I_1$;
- $K_1 = \mathrm{Ker}(\mathrm{GL}_2(\mathcal{O}_K) \to \mathrm{GL}_2(\mathbb{F}_q))$, $Z_1 = Z \cap K_1$;
- $(E, \mathcal{O}, \mathbb{F})$: for coefficients of representations.

Assumptions

Throughout the lecture, $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ will be reducible non-split

$$\bar{\rho} \cong \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1} p^i(r_i+1)} & * \\ 0 & 1 \end{pmatrix}$$

with $3 \leq r_i \leq p-6$. Set

$$\sigma_0 := (r_0, r_1, \ldots, r_{f-1})$$

called "ordinary" Serre weight. From Lecture 3, $\sigma_0 \in W(\bar{\rho})$.

Let $\pi_v(\bar{r})$ = admissible smooth $\mathbb{F}$-representation of G in mod p cohomology (cf. Lecture 1) with $\bar{r}|_{F_v} \cong \bar{\rho}$. Assume $r = 1$ (i.e. minimal case). Keep global technical conditions in Lecture 8.

Some useful facts

(1) $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi_{\sigma_0}) \cap W(\bar{\rho}) = \{\sigma_0\}$.

(2) let $\pi_0 := \langle G.\sigma_0 \rangle$, then π_0 is principal series and $\mathrm{soc}_G \pi_v(\bar{r}) = \pi_0$.

(3) **(Le)** $\pi_v(\bar{r})^{K_1} \cong D_0(\bar{\rho})$.

(4) If $\mathrm{Ext}^1_K(\sigma, \pi_v(\bar{r})) \neq 0$ for some Serre weight σ, then $\sigma \in W(\bar{\rho})$.

(5) **(H., Breuil-Ding)** $\mathrm{Ord}_B(\pi_v(\bar{r}))$ is semisimple (as T-rep.).

Remark. The point is that this excludes the following possibility :

$$(\pi_0 \text{ --- } \pi_0) \hookrightarrow \pi_v(\bar{r}).$$

Emerton has defined a functor of **ordinary parts** :

$$\mathrm{Ord}_B : \ \mathrm{Rep}_{\mathbb{F}}^{\mathrm{sm}}(G) \to \mathrm{Rep}_{\mathbb{F}}^{\mathrm{sm}}(T)$$

- an adjunction isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G \tau, \pi) \cong \mathrm{Hom}_T(\tau, \mathrm{Ord}_B \pi).$$

- $\mathrm{Ord}_B(\mathrm{Ind}_B^G \tau) \cong \tau$; $R^{f+1}\mathrm{Ord}_B(-) = 0$.

Let π^{ord} be the image of

$$\mathrm{Ind}_B^G \mathrm{Ord}_B \pi \to \pi.$$

Theorem 1 (H.-Wang)

The Gelfand-Kirillov dimension of $\pi_v(\bar{r})$ is f.

As in Lecture 6, can deduce that $\pi_v(\bar{r})^\vee$ is Cohen-Macaulay module of grade $2f$ (over $\Lambda := \mathbb{F}[\![I_1/Z_1]\!]$), and essentially self-dual.

Strategy of the proof :

(1) Show $[\pi_v(\bar{r})[\mathfrak{m}_{K_1}^2] : \sigma_0] = 1$.

(2) Show $[\pi_v(\bar{r})[\mathfrak{m}_{K_1}^2] : \sigma] = 1$ for any $\sigma \in W(\bar{\rho})$.

(2') Show $[\pi_v(\bar{r})[\mathfrak{m}_{I_1}^3] : \chi] = 1$ for any $\chi \in \pi_v(\bar{r})^{I_1}$.

(3) As in Lecture 6, deduce $\mathrm{GK}(\pi_v(\bar{r})) \leq f$ (need to use results of [BHHMS1]).
Conclude by Gee-Newton.

Steps (2), (2')

These steps are purely representation theoretic.

Proposition 2

Let $\bar{\rho}$ be as above, π be an admissible smooth $\mathbb{F}$-rep. of G satisfying :

(a) $\pi^{K_1} \cong D_0(\bar{\rho})$;

(b) if $\mathrm{Ext}^1_K(\sigma, \pi) \neq 0$ for some Serre weight σ, then $\sigma \in W(\bar{\rho})$;

(c) there exists one $\sigma_0 \in W(\bar{\rho})$ such that $[\pi[\mathfrak{m}^2_{K_1}] : \sigma_0] = 1$.

Then the following hold :

(i) $[\pi[\mathfrak{m}^2_{K_1}] : \sigma] = 1$ for any $\sigma \in W(\bar{\rho})$;

(ii) $[\pi[\mathfrak{m}^3_{I_1}] : \chi] = 1$ for any $\chi \in \pi^{I_1}$.

Proof. Recall : the diagram $(D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho}))$ is indecomposable by Lecture 3.

Define two sets as follows :

$$\Sigma_0 := \{\sigma \in W(\bar{\rho}) : \; [\pi[\mathfrak{m}_{K_1}^2] : \sigma] = 1\}$$

$$\Sigma_1 := \{\chi \in \pi^{I_1} : \; [\pi[\mathfrak{m}_{I_1}^3] : \chi] = 1\}.$$

Σ_1 is stable under the action of $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ (the one induced from $D_1(\bar{\rho})$).

- Use (a) (b) to show : if $\chi \in D_{0,\sigma}(\bar{\rho})^{I_1}$, then $\chi \in \Sigma_1$ if and only if $\sigma \in \Sigma_0$. Obtain a direct summand subdiagram. $\left(\bigoplus_{\chi \in \Sigma_1} \chi \hookrightarrow \bigoplus_{\sigma \in \Sigma_0} D_{0,\sigma}(\bar{\rho})\right)$
- By (c), this is nonzero, hence equals to the whole thing.

Step (1) : Show $[\pi_v(\bar{r})[\mathfrak{m}_{K_1}^2] : \sigma_0] = 1$.

Let $\Gamma := \mathrm{GL}_2(\mathbb{F}_q)$ so that $\mathbb{F}[\Gamma] \cong \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}$. Let

$$\widetilde{\Gamma} := \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]/\mathfrak{m}_{K_1}^2.$$

Let $\mathrm{Proj}_{\Gamma}\sigma_0$, resp. $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0$ be a projective envelope of σ_0 for Γ-representations, resp. $\widetilde{\Gamma}$-representations.
Have

$$\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0 \twoheadrightarrow \mathrm{Proj}_{\Gamma}\sigma_0.$$

Need to show

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}\big(\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0, \pi_v(\bar{r})\big) \stackrel{?}{=} 1.$$

Let M_∞ (and R_∞) be a *minimal* patching functor for $\bar{\rho}$ (cf. Lecture 8), e.g. take

$$M_\infty(-) := \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathbb{M}_\infty, -^d)^d$$

for a minimal patched module $\mathbb{M}_\infty$.

Recall that $\mathbb{M}_\infty/\mathfrak{m}_\infty \cong \pi_v(\bar{r})^\vee$, so we have

$$M_\infty(\Theta)/\mathfrak{m}_\infty \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta, \pi_v(\bar{r}))^\vee.$$

Equiv. to show

Theorem 3

The R_∞-module $M_\infty\big(\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0\big)$ is cyclic.

Recall gluing lemma 2 of Lecture 8 :

Given finite dim. $\mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)]\!]$-modules Θ_1, Θ_2 which admit a common quotient Θ_0, form the fiber product

$$\Theta_1 \times_{\Theta_0} \Theta_2.$$

Apply $M_\infty(-)$ to get

$$0 \to M_\infty(\Theta_1 \times_{\Theta_0} \Theta_2) \to M_\infty(\Theta_1) \times M_\infty(\Theta_2) \to M_\infty(\Theta_0) \to 0.$$

Assume both $M_\infty(\Theta_1)$, $M_\infty(\Theta_2)$ are cyclic R_∞-modules with annihilator I_1, I_2 (hence so is $M_\infty(\Theta_0)$ with annihilator I_0), then

$$M_\infty(\Theta_1 \times_{\Theta_0} \Theta_2) \text{ is cyclic} \iff I_1 + I_2 = I_0.$$

Roughly, we glue $\mathrm{Proj}_{\Gamma}\sigma_0$ with an ordinary part of $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0$:

- $\Theta_1 := \mathrm{Proj}_{\Gamma}\sigma_0$.

  Theorem (Le) The R_∞-module $M_\infty(\mathrm{Proj}_{\Gamma}\sigma_0)$ is cyclic.

- $\Theta_0 := \mathrm{Ind}_{B(\mathbb{F}_q)}^{\Gamma}\chi_{\sigma_0}$ (a quotient of Θ_1).

- $\Theta_2 :=$ **ordinary part** of $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0$.

  Fact. There exists a (unique) quotient Θ_2 of $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0$ such that :

$$0 \to \sigma_0^{\oplus f} \to \Theta_2 \to \Theta_0 \to 0.$$

Structure of Θ_i

Cyclicity of $M_\infty(\Theta_2)$

Lemma

If $\mathrm{JH}(\Theta_0) \cap \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi = \{\sigma_0\}$ and $\mathrm{Ord}_B(\pi)$ is semisimple, then the projection $\Theta_2 \twoheadrightarrow \sigma_0$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma_0, \pi) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta_2, \pi).$$

Roughly, if a morphism $\beta : \Theta_2 \to \pi$ does not factor through $\Theta_2 \to \sigma_0$, then $\langle G.\mathrm{Im}(\beta)\rangle$ will contain a non-split extension

$$\pi_0 — \pi_0$$

which contradicts **Fact** (5).

Proof of $I_1 + I_2 = I_0$

Can work locally : replace R_∞ with $R_{\bar{\rho}}$

- have an explicit description of I_0 (Fontaine-Laffaille) and I_1 (Le) ;
- the action of R_∞ on $M_\infty(\Theta_2)$ factors through $R_{\bar{\rho}}^{\mathrm{red}}$ (:=reducible deformation ring), i.e.

$$I^{\mathrm{red}} \subset I_2.$$

  Fact. For π locally admissible s.t. $\mathrm{JH}(\Theta_0) \cap \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi) = \{\sigma_0\}$, then

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta_2, \pi^{\mathrm{ord}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta_2, \pi).$$

- show $I^{\mathrm{red}} + I_2 = I_0$, by comparing tangent spaces.

Example. $R_{\bar{\rho}}$ (with fixed determinant) is isomorphic to $\mathcal{O}[\![X_i, Y_i, Z_i : 1 \leq i \leq f]\!]$.

- If $W(\bar{\rho}) = \{\sigma_0\}$,

$$I_0 = (\varpi, Y_i, Z_i), \quad I_1 = (\varpi, Y_i, Z_i(Z_i - p)), \quad I^{\mathrm{red}} = (\varpi, Z_i).$$

- If $|W(\bar{\rho})| = 2$, then for some j

$$I_1 = \left(\varpi, Y_j Z_j(Z_j - p), Z_i(Z_i - p); i \neq j\right).$$

Remark. Although $\Theta_1 \times_{\Theta_0} \Theta_2$ is smaller than $\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0$, the cyclicity of $M_\infty(\Theta_1 \times_{\Theta_0} \Theta_2)$ implies that of $M_\infty(\mathrm{Proj}_{\widetilde{\Gamma}}\sigma_0)$.

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The main result of this section is :

Theorem 4 (H.-Wang)

As a G-representation, $\pi_v(\bar{r})$ is generated by $D_0(\bar{\rho})$.

Corollary

We have $\mathrm{End}_G(\pi_v(\bar{r})) = \mathbb{F}$.

Proof. Taking restriction gives

$$\mathrm{End}_G(\pi_v(\bar{r})) \to \mathrm{End}_{\mathcal{DIAG}}\big(D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho})\big)$$

which is injective by **Theorem 3**. From Lecture 3, the target is isomorphic to $\mathbb{F}$. □

Example/Motivation

Take $f = 1$, so $\pi_v(\bar{r}) \cong (\pi_0 \text{ — } \pi_1)$. Let $\Omega \cong \mathrm{Inj}_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}\sigma_0$ together with a smooth action of G and assume $\pi_v(\bar{r}) \hookrightarrow \Omega$ (cf.[**BP**]).

Paškūnas : the G-socle filtration of Ω looks like :

$$\begin{array}{ll} \mathrm{soc}^2_G\Omega & \pi_v(\bar{r}) \oplus \pi_v(\bar{r}) \oplus \pi_v(\bar{r}) \\ \mathrm{soc}^1_G\Omega & \pi_v(\bar{r}) \oplus \pi_v(\bar{r}) \\ \mathrm{soc}^0_G\Omega & \pi_v(\bar{r}) \end{array}$$

Theorem : Let $\mathcal{E} := \mathrm{soc}^{\leq 1}_G\Omega$.

$$\begin{array}{ccccc} \mathrm{Ext}^1_G(\pi_v(\bar{r}), \pi_v(\bar{r})) & \longrightarrow & \mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma_0, \pi_v(\bar{r})) & \xleftarrow{\cong} & \mathrm{Ext}^1(\sigma_0, D_0(\bar{\rho})) \\ \uparrow & \nearrow_{\cong} & & & \\ \mathcal{E} & & & & \end{array}$$

Corollary : if $\pi \subset \Omega$ with $\pi[\mathfrak{m}_{K_1}^2]$ mulitiplicity free, then $\pi = \pi_v(\bar{r})$.

Proof. Note : $\pi = \pi_v(\bar{r})$ if and only if $\pi \cap \mathcal{E} = \pi_v(\bar{r})$.

If $\pi \cap \mathcal{E} \supsetneq \pi_v(\bar{r})$, then some non-split extension

$$[\pi_v(\bar{r}) — \pi_v(\bar{r})] \hookrightarrow \pi$$

which implies

$$[D_0(\bar{\rho}) — \sigma_0] \hookrightarrow \pi|_{\mathrm{GL}_2(\mathcal{O}_K)}.$$

Hence $[\pi[\mathfrak{m}_{K_1}^2] : \sigma_0] \geq 2$. □

The proof of Theorem 4

The starting point is :

Lemma 5

The G-cosocle of $\pi_v(\bar{r})$ is an irreducible PS, say π_f.

Proof.

- By **Fact** (2), $\mathrm{soc}_G \pi_v(\bar{r}) = \pi_0$ is PS ;
- As remarked, $\mathrm{GK}(\pi_v(\bar{r})) = f$ implies that $\pi_v(\bar{r})^\vee$ is essentially self-dual ;
- Recall Thm of Kohlhaase : $\mathrm{E}^{2f}(\pi_0^\vee)$ is again PS.

Criterion

Let $\tau \subset \pi_v(\bar{r})|_I$. If for some i, some $\chi : I \to \mathbb{F}^\times$, the composition

$$\begin{array}{ccc} \mathrm{Ext}^i_I(\chi, \tau) & & \\ \downarrow \beta_i & & \\ \mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) & \xrightarrow{\gamma_i} & \mathrm{Ext}^i_I(\chi, \pi_f) \end{array}$$

is non-zero, then $\pi_v(\bar{r})$ can be generated by τ as G-representation.

We will find some χ, i, τ such that "Criterion" applies.

How to choose χ, i and τ ?

Assume $W(\bar{\rho}) = \{\sigma_0\}$ for simplicity. Know the following information :

- $\mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) \neq 0$ if and only if $\chi \in \pi_v(\bar{r}))^{I_1}$ and

$$\dim_{\mathbb{F}} \mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) = \binom{2f}{i}.$$

  This suggests to take : $\chi = \chi_{\sigma_0}$ (the ordinary character).

- π_f has injective dimension $2f$, and

$$\dim_{\mathbb{F}} \mathrm{Ext}^i_I(\chi, \pi_f) = \begin{cases} 0 & i < f \\ \binom{f}{2f-i} & f \leq i \leq 2f \end{cases}$$

  This suggests to take $i = 2f$.

- The multiplicity-freeness of $\pi_v(\bar{r})[\mathfrak{m}_{I_1}^3]$ suggests : if take $\tau = \pi_v(\bar{r})[\mathfrak{m}_{I_1}^2]$ then

$$\dim_{\mathbb{F}} \mathrm{Ext}^1_I(\chi, \tau) = 2f$$

and the map

$$\beta_1 : \ \mathrm{Ext}^1_I(\chi, \tau) \to \mathrm{Ext}^1_I(\chi, \pi_v(\bar{r}))$$

is an isomorphism.

In summary, in the diagram of "Criterion"

$$\mathrm{Ext}^i_I(\chi, \tau) \xrightarrow{\beta_i} \mathrm{Ext}^i_I(\chi, \pi_v(\bar{r})) \xrightarrow{\gamma_i} \mathrm{Ext}^i_I(\chi, \pi_f)$$

take

- $\chi = \chi_{\sigma_0}$
- $i = 2f$
- $\tau =$ a variant of $\pi_v(\bar{r})[\mathfrak{m}^2_{I_1}]$.

Show

(1) γ_{2f} is an isomorphism (easier);

(2) β_{2f} is a surjection for any $0 \leq i \leq 2f$. Actually, inductively show β_i is surjective for any $0 \leq i \leq 2f$.

Step (2) : β_i is surjective

To deduce injectivity of β_{2f} from that of β_0, β_1, need :

Key ingredient : $\pi_v(\bar{r})^\vee|_I$ admits a Koszul complex projective resolution, as M_∞ is flat over R_∞ (which is regular) and $M_\infty/\mathfrak{m}_\infty \cong \pi_v(\bar{r})^\vee$.

Example. when $f=1$, $\pi_v(\bar{\rho}) = (\pi_0 — \pi_1)$, Paškūnas shows :

$$0 \to \Omega^\vee \overset{(-y,x)}{\to} \Omega^\vee \oplus \Omega^\vee \overset{\binom{x}{y}}{\to} \Omega^\vee \to \pi_v(\bar{r})^\vee \to 0.$$

Consider the following situation : $(R, \mathfrak{m})$ = noetherian local ring, $\underline{x} := (x_1, \ldots, x_n)$ with $x_i \in \mathfrak{m}$. Assume

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_2 & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{\beta}_2} & & \downarrow{\scriptstyle \widetilde{\beta}_1} & & \downarrow{\scriptstyle \widetilde{\beta}_0} & & \\ \cdots & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & 0 \end{array}$$

where

- $K_\bullet = K_\bullet(\underline{x}, R)$ = is Koszul complex, with $K_i \cong R^{\binom{n}{i}}$
- $F_\bullet$ = complex of free R-modules.

Lemma (Serre)

Assume

(a) $x_1, \ldots, x_n$ are linearly independent mod $\mathfrak{m}^2$;

(b) $\widetilde{\beta}_0 : K_0 \to F_0$ is a direct summand.

Then $\widetilde{\beta}_i : K_i \to F_i$ is a direct summand for all $0 \leq i \leq n$.

In practice, can not take $R = R_\infty$ in Serre's lemma, as R_∞ does not act on an injective resolution of τ.

To solve this, let $\lambda := (\mathrm{Proj}_I \chi^\vee)/\mathfrak{m}_{I_1^3}$ so that

$$\mathrm{End}_I(\lambda) \cong \mathbb{F}[x_i, y_i; 0 \le i \le f-1]/(x_i, y_i)^2.$$

Choose minimal projective resolutions :

$$Q_\bullet \twoheadrightarrow \tau^\vee, \quad K_\bullet \twoheadrightarrow \pi_v(\bar{r})^\vee$$

with $K_\bullet$ being Koszul, get morphisms $\mathrm{Hom}_I(K_\bullet, \lambda)^\vee \to \mathrm{Hom}_I(Q_\bullet, \lambda)^\vee$ of $\mathrm{End}_I(\lambda)$-modules, and

- Serre's lemma applies with $R := \mathrm{End}_I(\lambda)$. Actually get $\widetilde{\beta}_i$ are isomorphisms.
- $\mathbb{F} \otimes_R \mathrm{Hom}_I(Q_\bullet, \lambda)^\vee$ recovers $\mathrm{Hom}_I(Q_\bullet, \chi^\vee)^\vee$.

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Theorem 6

If $\bar{\rho}$ is reducible non-split, then

$$\pi_0 — \pi' — \pi_f$$

with π_0, π_f principal series. If moreover $f = 2$, then π' is supersingular.

Already known : the G-socle of $\pi_v(\bar{r})$ is π_0 and G-cosocle is π_f.

Assume $f = 2$. Need to show π' is irreducible and supersingular.

Proof.

(1) $\pi_v(\bar{r})/\pi_0$ contains (at least) one supersingular factor, using that "$\mathrm{Ord}_B(\pi_v(\bar{r}))$ is semi-simple" and

Lemma 1 ([BP,§8])

If π is (irred.) non-supersingular and $\pi \neq \pi_0$, then $\mathrm{Ext}^1_G(\pi, \pi_0) = 0$.

(2) Show that the G-socle of $\pi_v(\bar{r})/\pi_0$ is irreducible, say π_1. Namely,

$$(\pi_0 \text{ --- } \pi_1) \hookrightarrow \pi_v(\bar{r}).$$

Let Q be the quotient, **Need to show** : $Q \cong \pi_2$.

(3) Using **Theorem 4** to show Q can be generated by σ_2.

Lemma 2

Let Q be an admissible quotient of $I(\sigma_2) := \text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)Z}^{G}\,\sigma_2$. Assume the G-cosocle of Q is irreducible and isomorphic to

$$\pi_2 := I(\sigma_2)/(T-\lambda)$$

for some $\lambda \in \mathbb{F}^{\times}$. Then

$$Q \cong I(\sigma_2)/(T-\lambda)^n, \quad \text{some } n \geq 1.$$

(4) Show $n = 1$ by self-duality of $\pi_v(\bar{r})$.

Proof of Lemma 2. Let $V := \mathrm{Ker}(Q \twoheadrightarrow \pi_2)$. If $V \neq 0$ then we claim $\mathrm{Hom}_G(\pi_2, V) \neq 0$. By Barthel-Livné, $0 \to I(\sigma_2) \to I(\sigma_2) \to \pi_2 \to 0$, so

$$0 \to \mathrm{Hom}_G(\pi_2, V) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma_2, V) \overset{T-\lambda}{\to} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma_2, V).$$

If $\mathrm{Hom}_G(\pi_2, V) = 0$, then $T - \lambda$ is an isomorphism and get an injection

$$0 \to \mathrm{Ext}^1_G(\pi_2, V) \overset{\kappa}{\to} \mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma_2, V).$$

However, the (non-zero) extension class $0 \to V \to Q \to \pi_2 \to 0$ inside $\mathrm{Ext}^1_G(\pi_2, V)$ is sent to zero under κ, because Q is a quotient of $I(\sigma_2)$ (which induces $\sigma_2 \hookrightarrow Q|_{\mathrm{GL}_2(\mathcal{O}_K)}$). A contradiction.

The above argument still works for any nonzero quotient of Q, showing that any JH factor of Q is isomorphic to π_2. Conclude by Barthel-Livné.

Thank you !