p-adic LLC for $GL_2(\mathbb{Q}_p)$:

mod p:

$$\left.\begin{array}{l} \pi(r,0) : \text{supersingular with socle} \\ \mathrm{Sym}^r \mathbb{F}^2 \\ \mathrm{Sym}^{p-1-r}\mathbb{F}^2 \otimes \det^r \end{array}\right] \longleftrightarrow \mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}(\omega_2^{r+1})$$

$$\left.0 \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \chi_2 \otimes \chi_1 \omega^{-1} \to * \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)} \chi_1 \otimes \chi_2 \omega^{-1} \to 0 \right] \longleftrightarrow [0 \to \chi_1 \to \bar\rho \to \chi_2 \to 0]$$

if $\chi_1 \chi_2^{-1} \notin \{1, \omega\}$

socle: if $\chi_1 = \omega^{r+1}$: $\mathrm{Sym}^r \mathbb{F}^2$

$\chi_2 = 1$

$r \in \{1, \ldots, p-3\}$

& $\boxed{\text{also}}$ $\mathrm{Sym}^{p-3-r}\mathbb{F}^2 \otimes \det$

iff $*$ is split iff $\bar\rho$ is split

Serre weights:

$W(\bar\rho) = \{\mathrm{Sym}^r \mathbb{F}, \mathrm{Sym}^{p-3-r}\mathbb{F}^2 \otimes \det^{r+1}\}$

iff $\bar\rho$ is split

Char 0 (crystalline case)

$$k \geq 2,\ a_p \in \mathfrak{m}_E \rightsquigarrow \left\{ \begin{array}{l} \text{filtered } \varphi\text{-module } D_{k,a_p} \text{ over } E \\ \text{weakly admissible} \end{array} \right.$$

$$\Big\updownarrow \text{ Colmez-Fontaine}$$

V_{k,a_p} crystalline, HT weight $(0,k-1)$

$$WD(V_{k,a_p}) \simeq \mu_{\lambda_1} \oplus \mu_{\lambda_2}, \quad \lambda_1 \cdot \lambda_2 = p^{k-1}, \quad \lambda_1 + \lambda_2 = a_p$$

$$\text{Breuil-Berger } \Big\updownarrow$$

$$\left(\frac{\mathrm{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{GL_2(\mathbb{Q}_p)} \mathrm{Sym}^{k-2} E^2}{T - a_p} \right) \quad \Lambda$$

"Standard" lattice

Berger:

$$\left[\left(\frac{\operatorname{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{GL_2(\mathbb{Q}_p)} \operatorname{Sym}^{k-2}\mathcal{O}_E}{T-a_p}\right)\otimes_{\mathcal{O}_E}\mathbb{F}\right]^{ss} \longleftrightarrow \left(\overline{V_{k,a_p}}\right)^{ss}$$

$$= \pi(k-2,0) \text{ if } k \leq p+1$$