Multivariable $(\varphi, \mathcal{O}_K^\times)$-modules

p prime number K unramified extension of $\mathbb{Q}_p$ degree f.
$\mathbb{F}_q$ residue field. $\mathbb{F}/\mathbb{F}_q$ finite extension.

Convention: filtered abelian groups have increasing filtration $(F_n M)_{n \in \mathbb{Z}}$
$$\mathrm{gr}(M) = \bigoplus_{n \in \mathbb{Z}} \mathrm{gr}_n(M) \qquad \mathrm{gr}_n(M) = F_n M / F_{n-1} M.$$

1. Completed algebra of the pro-p-Iwahori

$I_1 = \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix} \subset GL_2(K) \qquad Z_1 = (1+p\mathcal{O}_K)\mathrm{Id}$ its center

$G = I_1 / Z_1$ pro-p-group.

$\rightsquigarrow \mathbb{F}[[G]]$ local complete noetherian algebra of maximal ideal $\mathfrak{m}_G$

Rk: π rep of $GL_2(K)$ with central character, $I_1 \curvearrowright \pi$. $\rightsquigarrow G \curvearrowright$

$\mathbb{F}[[G]]$ filtered $F_n = \begin{cases} \mathbb{F}[[G]] & n \geq 0 \\ \mathfrak{m}_G^{-n} & n \leq 0. \end{cases}$

Theorem (Clozel, Lazard) Assume $p > 2$

Then $\mathrm{gr}(\mathbb{F}[[G]]) \simeq \bigotimes_{j=0}^{f-1} U(\mathfrak{g}_j)$ with $\mathfrak{g}_j = \mathbb{F}y_j \oplus \mathbb{F}z_j \oplus \mathbb{F}h_j$

with relations $[y_j, z_j] = h_j$ + h_j central. $\deg(y_j) = \deg(z_j) = -1$

[Idea of proof: Lazard $\rightsquigarrow$ H pro-p-group with p-saturated valuation
$$\mathrm{gr}\,\mathbb{F}[[H]] \simeq U(\mathbb{F} \otimes_{\mathbb{Z}_p} \mathrm{Lie}\, H)$$
$\mathrm{Lie}\, G \simeq \bigoplus \mathfrak{g}_j$

$y_j \longleftrightarrow \sigma_j \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad z_j \longleftrightarrow \sigma_j \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}$

$h_j \longleftrightarrow \sigma_j \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$]

$J \subset \mathrm{gr}\,\mathbb{F}[[G]]$ (left) ideal generated by $h_j, y_j z_j, j = 0 \dots f-1$.

$\hookrightarrow$ 2-sided and $\mathrm{gr}\,\mathbb{F}[[G]] / J \simeq \underbrace{\bigotimes_{j=0}^{f-1} \mathbb{F}[y_j, z_j] / (y_j z_j)}_{f\text{-dim complete intersection}}$

$\pi \in \mathrm{Rep}^{adm}_{\mathbb{F}} GL_2(K)$ with central char
$\hookrightarrow \pi^\vee$ f g $\mathbb{F}[[G]]$-module

$gr(\pi^\vee) = \bigoplus_{n \leq 0} gr_n(\pi^\vee) \qquad gr_n(\pi^\vee) = \frac{m_G^{-n}\pi^\vee}{m_G^{-n+1}\pi^\vee}$.

$\hookrightarrow$ f g $gr\,\mathbb{F}[[G]]$-module

<u>Def</u> $\mathcal{C}_1$ category of π as above st $\exists\, m \geq 0$, $J^m gr(\pi^\vee) = 0$.

<u>Remarks</u> 1) $\mathcal{C}_1$ abelian category.

2) $K = \mathbb{Q}_p$ $\quad \mathcal{C}_1 \supset$ irreducible rep.

3) $\pi \in \mathcal{C}_1 \implies GK\dim(\pi) \leq f$.

[Krull dim $\left(gr\,\mathbb{F}[[G]]/J \right) \leq f \implies j_{gr\mathbb{F}[[G]]}(gr(\pi^\vee)) \leq f$ $\Leftarrow$ Björk

well defined
for modules over
enveloping algebras

$\Downarrow$

$GK\dim(\pi) \leq f$]

<u>Thm</u> With notation and assumptions of lecture 1,
$\pi_v(\bar{r}) \in \mathcal{C}_1$ $\rightsquigarrow$ genericity + Taylor Wiles.

$\implies \mathcal{C}_1$ contains enough objects of global origin.

2. $\mathcal{C}_1 \longrightarrow (\varphi, \mathcal{O}_K^\times)$-modules

$\underbrace{\mathbb{F}[[\begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix}]]}_{N_0} \simeq \mathbb{F}[[Y_0, \dots, Y_{f-1}]] \qquad Y_i = \sum_{a \in \mathbb{F}_q^\times} a^{-p^i} \begin{pmatrix} 1 & [a] \\ 0 & 1 \end{pmatrix}$

$m_{N_0} = (Y_0, \dots, Y_{f-1}) \qquad y_i := Y_i \bmod m_{N_0}^2$

$gr(R) \simeq \mathbb{F}[y_0, \dots, y_{f-1}]$.

$v : \mathbb{F}[[N_0]] \longrightarrow \mathbb{Z} \cup \{+\infty\}$ $\quad m_{N_0}$-adic filtration $(v(x) = \sup\{ i \mid x \in m_{N_0}^i \})$

$v(xy) = v(x) + v(y) \rightsquigarrow v_S : \mathbb{F}[[N_0]]_S \longrightarrow \mathbb{Z} \cup \{+\infty\}$

where $S = \{ (Y_0 \dots Y_{f-1})^n \mid n \in \mathbb{N} \}$.

$A := \widehat{\mathbb{F}[[N_0]]_S}$ $\quad$ completion for the topology induced by v_S.

$v_A : A \longrightarrow \mathbb{Z} \cup \{+\infty\}$. $\quad$ A is a filtered ring:

$F_n A = v_A^{-1}([-n, +\infty])$

Additional structures on A: $\begin{pmatrix} p & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1}$ on N_0

$\rightsquigarrow$ $\varphi : \mathbb{F}[[N_0]] \hookrightarrow \mathbb{F}[[N_0]]$ finite flat ring endomorphism. $\varphi(Y_1) = Y_{i-1}^p$

$\rightsquigarrow$ extends to $\varphi : A \hookrightarrow A$ (finite flat of degree p^t).

$a \in \mathcal{O}_K^\times$, $\begin{pmatrix} a & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} a & \\ & 1 \end{pmatrix}^{-1}$ on N_0

$\rightsquigarrow$ action of $\mathcal{O}_K^\times$ on $\mathbb{F}[[N_0]] \rightsquigarrow$ cts action of $\mathcal{O}_K^\times$ on A.

The actions of φ and $\mathcal{O}_K^\times$ commute.

<u>Def</u> A $(\varphi, \mathcal{O}_K^\times)$-module (over A) is a f g A-module D with

$$\left.\begin{array}{l} \varphi : D \longrightarrow D \quad \varphi\text{-semilinear} \\ \mathcal{O}_K^\times \longrightarrow \mathrm{Aut}(D)_{\text{cts}} \quad \mathcal{O}_K^\times\text{-semilinear} \end{array}\right) \text{ commute}$$

We say that D is <u>étale</u> if $1 \otimes \varphi : A \otimes_{A,\varphi} D \longrightarrow D$ is an iso.

<u>Proposition</u>: Let M be a f g A-module + $\mathcal{O}_K^\times \longrightarrow \mathrm{Aut}(M)$ cts semilinear action. Then M is projective.

Based on <u>lemma</u> The $\mathcal{O}_K^\times$-stable ideals of A are 0 and A.

[follows closely a strategy of Ardakov-Wadsley.

$*$ $\mathfrak{a} \subset A$ $\mathcal{O}_K^\times$-stable $\rightsquigarrow$ $\mathfrak{a} = A \cdot (\mathfrak{a} \cap \varphi(A))$

$*$ $\mathfrak{a} \neq A$, $\mathfrak{a} = \bigcap_{n \geq 0} A \cdot (\mathfrak{a} \cap \varphi^n(A)) = \{0\}$]

For the proposition:

Biduality exact sequence $\underbrace{E_2^{p,q}}_{\text{torsion if } p \text{ or } q > 0} = \mathrm{Ext}_A^p(\mathrm{Ext}_A^{-q}(M, A), A) \Rightarrow M$.

$\mathcal{O}_K^\times \curvearrowright M$! fg $\qquad \circlearrowleft \mathcal{O}_K^\times$

$\Rightarrow \mathrm{Ann}(E_2^{p,q}) = A$ for p, or $q > 0$.

$\Rightarrow M \simeq (M^\vee)^\vee$ + $\mathrm{Ext}_A^p(M^\vee, A) = 0$ for $p > 0$.

A has finite global dimension $\Rightarrow M^\vee$ proj $\Rightarrow M$ proj $\square$

<u>Specialization</u>:

$$\begin{array}{ccc} N_0 & \xrightarrow{\mathrm{Tr}} & \mathbb{Z}_p \\ \wr\!\! & & \\ \mathcal{O}_K & & \end{array} \qquad \begin{array}{ccc} \mathbb{F}[[N_0]] & \xrightarrow{\mathrm{Tr}} & \mathbb{F}[[\mathbb{Z}_p]] \simeq \mathbb{F}[[X]] \\ \cap & & \cap \\ A & \xrightarrow{\mathrm{Tr}} & \mathbb{F}((X)) \end{array}$$

commutes to φ and $\mathbb{Z}_p^\times \subset \mathcal{O}_K^\times$.

D $(\varphi, \mathcal{O}_K^\times)$-module $\rightsquigarrow D \otimes_{A,\mathrm{Tr}} \mathbb{F}((X))$ is a (φ, Γ)-module

étale if D is.

Theorem There exists an exact functor
$$D_A^{et} : \mathcal{C}_1 \longrightarrow \left(\text{étale } (\varphi, \mathcal{O}_K^\times)\text{-modules} \right) \text{ such that}$$
$$D_A^{et} \otimes_{A, T_1} \mathbb{F}((X)) \simeq D_\xi^\vee|_{\mathcal{C}_1}$$

Corollary $\forall \pi \in \mathcal{C}_1, \quad \dim_{\mathbb{F}((X))} D_\xi^\vee(\pi) < \infty.$

Construction: $\pi \in \mathcal{C}_1$. $\pi^\vee$ filtered $\mathbb{F}[\![G]\!]$-module

$N_0 \hookrightarrow G \longrightarrow \pi^\vee$ is a filtered $\mathbb{F}[\![N_0]\!]$-module

tensor product filtration on $\pi_S^\vee = \mathbb{F}[\![N_0]\!]_S \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$

$$F_m(\pi^\vee) = \varinjlim_{s \gg 0} (Y_0 \cdots Y_{f-1})^{-s}\, m_G^{sf-m}\, \pi^\vee.$$

$$D_A(\pi) = \widehat{\pi_S^\vee} = \varprojlim_m \pi^\vee / F_m \pi^\vee \quad \left(= A \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee \right)$$

$\rightsquigarrow$ cts + semilinear action of $\mathcal{O}_K^\times$ on $D_A(\pi)$

Pb: naive def of φ on $\pi_S^\vee$ is not continuous with respect to the tensor product filtration.

define ψ $\quad \psi : \pi^\vee \xrightarrow{\sim} \pi^\vee \qquad \psi(\varphi(a)v) = a\psi(v)$

$$\lambda \longmapsto \lambda \circ (p_1)$$

$$\rightsquigarrow \pi_S^\vee \xrightarrow{\psi} \pi_S^\vee$$
$$\frac{x}{(Y_0 \cdots Y_{f-1})^{pm}} \longmapsto \frac{\psi(x)}{(Y_0 \cdots Y_{f-1})^m}$$

Fact ψ continuous with respect to the filtration of $\pi_S^\vee$

[ψ decreases denominators]

$\rightsquigarrow \psi : D_A(\pi) \longrightarrow D_A(\pi)$ and ψ commutes to $\mathcal{O}_K^\times$

linearization $\beta : D_A(\pi) \longrightarrow A \otimes_{A,\varphi} D_A(\pi)$ $\quad$) $\mathcal{O}_K^\times$-equivariant.

$$v \longmapsto \sum_{g \in N_0/N_0^p} \delta_g \otimes_\varphi \psi(\delta_g^{-1} v)$$

If β is an iso, can define φ such that $1 \otimes \psi = \beta^{-1}$

$$D_A^{et}(\pi) = \varinjlim_{1 \otimes_{\varphi^m} \beta} A \otimes_{A,\varphi^m} D_A(\pi)$$

We obtain $\beta^{et}: D_A^{et}(\pi) \xrightarrow{\sim} A \otimes_{\Lambda,\varphi} D_A^{et}(\pi)$

$$\begin{array}{ccc} \| & & \| \\ \varinjlim_n A \otimes_{\varphi^n} D_A(\pi) & \xrightarrow[(1\otimes_{\varphi^n}\beta)]{} & \varinjlim_n A \otimes_{\varphi^{n+1}} D_A(\pi) \end{array}$$

$D_A^{et}(\pi)$ has a $(\varphi, \mathcal{O}_K^\times)$-structure (but still can be infinitely generated).

<u>Proposition</u>: $D_A(\pi)$ is a f.g. A-module and $D_A^{et}(\pi)$ is a quotient of $D_A(\pi)$.

[Assume for simplicity that $J\, gr(\pi^\vee) = 0$.

From the definition of the filtration on $\pi_S^\vee$, we have

$$gr(D_A(\pi)) \simeq gr(A) \otimes_{gr\,\mathbb{F}[[N_0]]} gr(\pi^\vee) \simeq gr(A) \otimes_{gr\,\mathbb{F}[[N_0]]} gr\,\mathbb{F}[[G]]/J \underbrace{\otimes_{gr\,\mathbb{F}[[G]]} gr(\pi^\vee)}_{f.g.}$$

$gr\,\mathbb{F}[[N_0]] \hookrightarrow gr\,\mathbb{F}[[G]]/J \simeq \mathbb{F}[y_0, z_0, \dots, y_{f-1}, z_{f-1}]/(y_j z_j)$

$\wr\|$

$\mathbb{F}[y_0, \dots, y_j]$

$y_j \mapsto y_j$

$$\Rightarrow gr(A) \otimes_{gr\,\mathbb{F}[[N_0]]} gr\,\mathbb{F}[[G]]/J \simeq \mathbb{F}[y_0, \dots, y_{f-1}, (y_0 \cdots y_{f-1})^{-1}] = gr(A)$$

$\Rightarrow gr(D_A(\pi))$ f.g. $gr(A)$-module $\underset{\text{complete}}{\Rightarrow} D_A(\pi)$ f.g. A-module.

Second part. Note

$$\begin{tikzcd} D_A(\pi) \arrow[rr, "\beta_n"] \arrow[dr, "\beta_m"'] & & A \otimes_{A,\varphi^n} D_A(\pi) \\ & A \otimes_{A,\varphi^m} D_A(\pi) \arrow[ur, "1 \otimes_{\varphi^m} \beta_{n-m}"'] & \end{tikzcd} \quad \text{for } m \leq n.$$

$\operatorname{Ker} \beta_n \supset \operatorname{Ker} \beta_m$ $\quad N = \bigcup_{n \geq 0} \operatorname{Ker} \beta_n$ $\quad \beta(N) \subset A \otimes_{\varphi,A} N \subset A \otimes_{A,\varphi} D_A(\pi)$.

$\rightsquigarrow \beta : D_A(\pi)/N \hookrightarrow A \otimes_{A,\varphi} (D_A(\pi)/N)$ injective

$\circlearrowleft \mathcal{O}_K^\times \qquad \circlearrowleft \mathcal{O}_K^\times$

$\operatorname{Ann}(\operatorname{Coker} \beta) \subset A$, $\ne 0$, $\circlearrowleft \mathcal{O}_K^\times$ $\Rightarrow \operatorname{Coker} \beta = 0$.

$$\rightsquigarrow \varinjlim A \otimes_{A,\varphi^n} D_A(\pi) = \varinjlim_n A \otimes_{A,\varphi^n} D_A(\pi)/N \simeq D_A(\pi)/N. \quad \square$$

<u>Remark</u>: Explicit bound on $\mathrm{rk}_A D_A^{et}(\pi) \leq \mathrm{rk}_A D_A(\pi)$.

$D_A(\pi)$ projective A-module.

<u>Lemma</u>: M f.g. filtered A-module such that $gr(M)$ is a f.g. $gr(A)$-module.

Then $\quad \dim_{Fr(A)} M \otimes \operatorname{Frac}(A) = \dim_{Fr(gr\,A)} gr\,M \otimes Fr(gr\,A)$.

$$\Rightarrow \mathrm{rk}_A D_A^{et}(\pi) \leq \mathrm{rk}_{gr\,A}\, gr\, D_A(\pi^\vee).$$

Assume that $J \cdot gr(\pi^\vee) = 0$. Let $\mathfrak{p}_0 = (z_0, \ldots, z_{f-1})$ minimal ideal of $gr\, \mathbb{F}[\![G]\!]/J$. Then $rk_{gr A}\, gr\, D_A(\pi) = \dim_{k(\mathfrak{p}_0)} (gr(\pi^\vee) \otimes k(\mathfrak{p}_0))$

More generally we define $m_{\mathfrak{p}_0}(\pi) = \sum_{n \geq 0} \dim_{k(\mathfrak{p}_0)} \left(\frac{\mathfrak{p}_0^n gr(\pi^\vee)}{\mathfrak{p}_0^{n+1} gr(\pi^\vee)} \right) \otimes k(\mathfrak{p}_0)$

<u>Theorem</u> $rk_A\, D_A^{et}(\pi) \leq rk_A\, D_A(\pi) = m_{\mathfrak{p}_0}(\pi)$.

Still have to prove

- exactness: $\underbrace{\pi \longmapsto \pi^\vee \longmapsto \pi^\vee_S \longmapsto D_A(\pi) \longmapsto D_A^{et}(\pi) = \varinjlim A \otimes_{A, \varphi^n} D_A(\pi)}_{\text{all exact functors}}$, $\widehat{\pi^\vee_S} = D_A(\pi)$

- $D_A^{et} \otimes_{A, T_A} \mathbb{F}((X)) \xrightarrow{\sim} D_\xi^\vee$.

construct the map: if $M \subset \pi^{N_1}$ $\mathbb{Z}_p^\times$ f.g. admissible $\mathbb{F}[\![\mathbb{Z}_p]\!][F]$-module.

$\pi^\vee \twoheadrightarrow M^\vee$ (cts) $\quad \pi^\vee_S \twoheadrightarrow M^\vee[X^{-1}]$

$D_A(\pi) \twoheadrightarrow M^\vee[X^{-1}]$ (complete)

$$\begin{array}{ccccc}
D_A(\pi) & \longrightarrow & D_A(\pi) \otimes_A \mathbb{F}((X)) & \xrightarrow{\otimes_K^\times} & M^\vee[X^{-1}] \\
\downarrow \beta & & \downarrow \beta \otimes 1 & D_A^{et}(\pi) & \downarrow \wr \quad (1 \otimes \varphi)^{-1} \\
A \otimes_{A, \varphi} D_A(\pi) & \longrightarrow & \mathbb{F}((X)) \otimes_\varphi (\mathbb{F}((X)) \otimes_A D_A(\pi)) & \longrightarrow & \mathbb{F}((X)) \otimes_\varphi M^\vee[X^{-1}].
\end{array}$$

$D_A^{et}(\pi) \twoheadrightarrow M^\vee[X^{-1}]$. $\rightsquigarrow$ $D_A^{et}(\pi) \longrightarrow D_\xi^\vee(\pi)$

Surjectivity: $\dim_{\mathbb{F}((X))} M^\vee[X^{-1}]$ for M as above bounded

$\Rightarrow$ $D_\xi^\vee(\pi) = M^\vee[X^{-1}]$ for some M

$\Rightarrow$ surjectivity.

<u>Corollary</u> $\pi \in \mathcal{C}_1$, $\dim_{\mathbb{F}((X))} D_\xi^\vee(\pi) \leq m_{\mathfrak{p}_0}(\pi)$.