Serre weights and diagrams for $Gl_2(\mathbb{Q}_{p^f})$

(a survey of [B.- Paškūnas])

$K = \mathbb{Q}_{p^f}$, $\mathbb{F}$ = coef. field = finite ext. of $\mathbb{F}_p$

① Serre weights of BDJ

↳ Buzzard · Diamond · Jarvis

② Weight cycling and multiplicity 1

③ Diagrams.

<u>Not for specialists</u>

① (i) F = tot. real field

D/F = quaternion algebra split at all places above p

split at <u>one</u> ∞ place

(ii) same but D is definite at all ∞ places

(iii) F^+ = tot. real field

F = tot imag quadr. ext. of F^+

$$\begin{tikzcd} & \text{Gl_2 at places above p} \\ G/_{F^+} = \text{unitary group} \arrow[ur] \arrow[dr] & \\ & \text{U(n) at ∞ place} \end{tikzcd}$$

$U_2(K)$ at v-pla

$v \mid p$ in $\left| \begin{array}{l} F \\ F^+ \end{array} \right.$, $K = \left| \begin{array}{l} F_v \\ F_v^+ \end{array} \right.$

$\bar{r} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_2(\mathbb{F})$ $\subseteq$ abs. irred.

$U^v =$ c. o. s/gp in $\left| \begin{array}{l} (D \otimes_F \mathbb{A}_F^{\infty,v})^\times \\ G(\mathbb{A}_{F^+}^{\infty,v}) \end{array} \right.$

$$\Pi_v(\bar{r}) := \begin{cases} \text{(i)} \ \mathrm{Hom}_{\mathrm{Gal}(\bar{F}/F)}\left(\bar{r}, \varinjlim_{U_v} H^1_{et}\left(X_{U^vU_v} \times_F \bar{F}, \mathbb{F}\right)\right) \\ \text{(ii)} \left(\varinjlim_{U_v} \left\{ f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times / U^v U_v \longrightarrow \mathbb{F} \right\}\right)[\mathfrak{m}_{\bar{r}}] \\ \text{(iii)} \left(\varinjlim_{U_v} \left\{ f : G(F^+) \backslash G(\mathbb{A}_{F^+}^\infty) / U^v U_v \longrightarrow \mathbb{F} \right\}\right)[\mathfrak{m}_{\bar{r}}] \end{cases}$$

$$\Pi_v(\bar{r}) \neq 0$$

$\Pi_v(\bar{r}) =$ smooth <u>admissible</u> repr of $GL_2(K) / \mathbb{F}$

$\Pi_v(\bar{r})^{K_1}$ finite dim. $/ \mathbb{F}$ $\quad K_1 = 1 + p M_2(\mathcal{O}_K)$

$\Rightarrow$ $\mathrm{soc}_{GL_2(\mathcal{O}_K)} \Pi_v(\bar{r})$ is finite dim.

$\Rightarrow$ direct sum of finitely many Serre weights

$(GL_2(\mathcal{O}_K) \to GL_2(\mathbb{F}_q$

BDJ $\Rightarrow$ give list of these Serre weights

$\bar\rho := \bar r_v = \bar r|_{\mathrm{Gal}(\bar K/K)}$ generic as follows:

- $\bar\rho$ reducible

inertia $\bar\rho|_{I_K} \cong \begin{pmatrix} \omega_f^{r_0+1+p(r_1+1)+\dots+p^{f-1}(r_{f-1}+1)} & * \\ 0 & 1 \end{pmatrix}$ (twist)

where $r_i \in \{0, \dots, p-3\}$

$(r_i) \neq (0,\dots,0)\, , \,(p-3,\dots,p-3)$

- $\bar\rho$ irreduc.

$\bar\rho|_{I_K} \cong \begin{pmatrix} \omega_{2f}^{r_0+1+\dots+p^{f-1}(r_{f-1}+1)} & 0 \\ 0 & \omega_{2f}^{q(\text{same})} \end{pmatrix}$ (twist)

where $r_i \in \{0, \dots, p-3\}$ if $i \neq 0$

$r_0 \in \{1, \dots, p-2\}$

(need to fix $\mathbb{F}_{p^{2f}} \hookrightarrow \mathbb{F}$)

Notation for Serre weights: $s_j \in \{0,\dots,p-1\}$

$j \in \{0,\dots,f-1\}$

$\theta : \mathcal{O}_K^\times \longrightarrow \mathbb{F}^\times$

$(s_0,\dots,s_{f-1}) \otimes \theta :=$

$\left(\mathrm{Sym}^{s_0}(\mathbb{F}^2) \otimes_{\mathbb{F}} \left(\mathrm{Sym}^{s_1}(\mathbb{F}^2)\right)^{(\mathrm{Fr})} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \left(\mathrm{Sym}^{s_{f-1}}(\mathbb{F}^2)\right)^{\mathrm{Fr}^{f-1}}\right)$

$\otimes$ Ooder

where $(\ )^{Fr^j}$: $GL_2(\mathbb{F}_q)$ acts by

$$x \in \mathbb{F}_q \longmapsto x^{p^j} \hookrightarrow \mathbb{F}$$

<u>Set of S.W. associated to</u> $\bar{\rho}|_{I_K}$ (BDJ).

denoted $W(\bar{\rho})$

- if $\bar{\rho}$ is reducible

$W(\bar{\rho}) := \{ (s_0, \dots, s_{f-1}) \otimes \Theta$ s.t.

$\exists\ J \subseteq \{0, \dots, f-1\}$ with

$$\bar{\rho}|_{I_K} \simeq \begin{pmatrix} \omega_f^{\sum_{j \notin J}(s_j+1)p^j} & * \\ 0 & \omega_f^{\sum_{j \in J}(s_j+1)p^j} \end{pmatrix} \otimes \Theta$$

where $*$ is Fontaine-Laffaille $\}$

- $\bar{\rho}$ irred. $W(\bar{\rho}) := \{ (s_0, \dots, s_{f-1}) \otimes \Theta$ s.t.

$\exists J$ with $\bar{\rho}|_{I_K} \simeq \begin{pmatrix} \omega_{2f}^{\sum_{j \notin J}(s_j+1)p^j + q\left(\sum_{j \in J}(s_j+1)p^j\right)} & 0 \\ 0 & \omega_{2f}^{q(\text{same})} \end{pmatrix} \otimes$

<u>Ex:</u> $f=2$, $\bar{\rho}$ reducible

$$\begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & * \end{pmatrix} \simeq \begin{pmatrix} \omega_2^{r_0+2} & * \end{pmatrix} \otimes \omega^{p-1+pr_1}$$

$$\begin{pmatrix} \omega_2 & * \\ 0 & 1 \end{pmatrix} \cong \begin{pmatrix} * & * \\ 0 & \omega_2^{p(p-1-r_1)} \end{pmatrix} \otimes \omega_2$$

always FL

$$\cong \begin{pmatrix} \omega_2^{p(r_1+2)} & * \\ 0 & \omega_2^{p-1-r_0} \end{pmatrix} \otimes \omega_2^{r_0+p(p-1)}$$

it can be FL

$$\cong \begin{pmatrix} 1 & * \\ 0 & \omega_2^{p-2-r_0+p(p-2-r_1)} \end{pmatrix} \otimes \omega_2^{r_0+1+p(r_1+1)}$$

FL $\Leftrightarrow$ $* = 0$

$$\Rightarrow W(\bar{\rho}^{ss}) = \{ (r_0,r_1),\ (r_0+1,\ p-2-r_1)\otimes \det^{p-1+pr_1},\ (p-2-r_0,\ r_1+1)\otimes \det^{r_0+p(p-1)},\ (p-3-r_0,\ p-3-r_1)\otimes \det^{r_0+1+p(r_1+1)} \}$$

$$|W(\bar{\rho})| \in \{1, 2, 4\}$$

$$(4 \Leftrightarrow \bar{\rho} = \bar{\rho}^{ss})$$

<u>Several facts about $W(\bar{\rho})$</u> :

- $(r_0, \dots, r_{f-1})$ always in $W(\bar{\rho})$
- $(p-3-r_0, \dots, p-3-r_{f-1}) \otimes \det^{\sum (r_j+1)p^j}$ is in $W(\bar{\rho})$ iff $\bar{\rho} = \bar{\rho}^{ss}$ <u>reduc.</u>
- if $\bar{\rho}$ is semi-simple, all S.w in $W(\bar{\rho})$ have the following form:

$(\dots, r_{i-1},\ p-2-r_i,\ p-3-r_{i+1}, \dots,\ p-3-r_{i+l},\ r_{i+l+1}+1,\ r_{\dots},$

p-2-, p-3-, … f), …)

(if $\bar\rho$ is irred. $\begin{matrix} p-3-r_0 \\ r_0+1 \end{matrix} \rightsquigarrow \begin{matrix} p-1-r_0 \\ r_0-1 \end{matrix}$). ⊗ ded

$|W(\bar\rho)| = 2^f$

$\sigma \in W(\bar\rho)$, length(σ) = "count how many p-2- and p-3- you have"

$\in \{0, \dots, f\}$.

if $\bar\rho$ is red non split: $|W(\bar\rho)| = 2^d$ for $d \in \{0, \dots, f-1\}$.

Theorem A (Barnet-Lamb, Gee, Geraghty, Kisin, Liu, Savitt, …)

Assume $p > 5$ is unramified in F +

$\bar r|_{G(\bar F / F(\zeta_p))}$ = abs. irred

+ further technical assumpt in set up (iii) (G/F^+ quasi-split at finite places…)

Then $W(\bar\rho)$ is exactly the set of S_w (up to mult.) in $\mathrm{soc}_{G_n(\mathcal{O}_n)} \pi_v(\bar r)$.

② Weight cycling and multiplicity 1

notation

$$\begin{array}{l} I = \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} = \text{Iwahori} \\ \cup\!\mid \\ I_1 = \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix} = \text{pro-}p\text{-Iwahori} \end{array}$$

$$I/I_1 = \begin{pmatrix} \mathbb{F}_q^\times & 0 \\ 0 & \mathbb{F}_q^\times \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \Big|^{I_1}_{I} = {}^{I_1}_{I}\Big| \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

$$\Rightarrow \underline{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \text{ respects } \pi_v(\bar{r})^{I_1}}$$

weight cycling is a way to relate the various $\sigma^{I_1} \hookrightarrow \pi_v(\bar{r})^{I_1}$ for $\sigma \in W(\bar{\rho})$

$(\dim_{\mathbb{F}} \sigma^{I_1} = 1)$

(Buzzard when $K = \mathbb{Q}_p$)

$$\begin{tikzcd} I \arrow[r, "\chi"] \arrow[dr, two heads] & \mathbb{F}^\times \\ & I/I_1 \arrow[u] \end{tikzcd}$$

$$\Rightarrow \chi\underbrace{\begin{pmatrix} a & b \\ px & d \end{pmatrix}}_{\in I} = \chi_1(a)\chi_2(d) \qquad \chi_i : \mathcal{O}_K^\times \to \mathbb{F}^\times$$

$$\chi = \chi_1 \otimes \chi_2$$

$$\chi^s := \chi \otimes \chi = \chi\left(\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix}^{-1}\right) \;|||$$

$$\lambda \cdot - \lambda_2 \otimes \lambda_1 - \lambda\left(\begin{pmatrix} & \\ p & 0\end{pmatrix}\cdot\begin{pmatrix} & \\ p & 0\end{pmatrix}\right) \;|||$$

$\sigma \hookrightarrow \pi_v(\bar{r})$, χ = action of I on σ^{I_1}

then I acts by χ^s on $\begin{pmatrix}0 & 1\\ p & 0\end{pmatrix}\sigma^{I_1}$ $(\subseteq \pi_v(\bar{r}$

$$\begin{tikzcd} \operatorname{ind}_I^{GL_2(O_n)} \chi^s \arrow[r, two heads] \arrow[d, phantom, "\wr|"] & \left\langle GL_2(O_n)\cdot\begin{pmatrix}0 & 1\\ p & 0\end{pmatrix}\sigma^{I_1}\right\rangle \;] \\ \operatorname{ind}_{B(\mathbb{F}_q)}^{GL_2(\mathbb{F}_q)} \chi^s & \end{tikzcd}$$

$\hookrightarrow$ Frobenius reciprocity

(= pr. series for $GL_2(\mathbb{F}_q)$)

Lemma 1 (BP): There is a (unique) smallest non zero quotient of $\operatorname{ind} \chi^s$ with socle = a Serre weight in $W(\bar{\rho})$.

Let $\delta(\sigma)$ be the Serre weight of this lemma

$$\sigma \rightsquigarrow \delta(\sigma) \rightsquigarrow \delta^2(\sigma) \rightsquigarrow$$

Ex: $f=2$, $\bar{\rho}$ red. split

$$\begin{tikzcd} (r_0, r_1) \arrow[loop below] & (p-2-r_0, r_1+1) \arrow[r, bend left] & (r_0+1, p-2- \arrow[l, bend left] \\ & & (p-3-r_0, p-3-r_1) \arrow[loop below] \end{tikzcd}$$

$\bar{\rho}$ semi-simple, $\exists\, n \geq 1$ s.t. $\delta^n(\sigma) = \sigma$

$\Rightarrow$ δ gives a partition of $W(\bar{\rho})$

Question : | can one "see" these cycles on $\bar\rho$?

Answer : | Yes! On the tensor induction

$$\mathrm{ind}_K^{\otimes\mathbb{Q}_p}\bar\rho := \bigotimes_{\tau\in\mathrm{Gal}(K/\mathbb{Q}_p)} \bar\rho(\tau\cdot\tau^{-1}) \quad \underset{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}{\circlearrowright}$$

$$\left(\mathrm{ind}_K^{\otimes\mathbb{Q}_p}\bar\rho\right)\Big|_{I_{\mathbb{Q}_p}} = \bigoplus \eta \longrightarrow \text{fund. char.}$$

δ-cycles have same size as the Frobenius cycles $\eta\mapsto\eta^p\mapsto\eta^{p^2}\cdots$

Suggested $\rightsquigarrow$. $\sigma\rightsquigarrow\delta(\sigma)$ right in c.

- $\sigma\in W(\bar\rho)$ should only appear in $\mathrm{soc}_{GL_2(\mathcal{O}_L)}(\pi_v(\bar r))$, NOT on $\dfrac{\pi_v(\bar r)^{K_1}}{\mathrm{soc}_{GL_2(\mathcal{O}_L)}\pi_v(\bar r)}$.

Prop (BP): ① ∃ a unique f. dim'l representation $D_0(\bar\rho)$ of $GL_2(\mathbb{F}_q)/\mathbb{F}$ such that:

(i) $\mathrm{soc}\, D_0(\bar\rho) = \bigoplus_{\sigma\in W(\bar\rho)}\sigma$

(ii) $\sigma\in W(\bar\rho)$ appears with mult 1 in

$D_0(\rho)$

(iii) $D_0(\bar\rho)$ is max. for (i) + (ii).

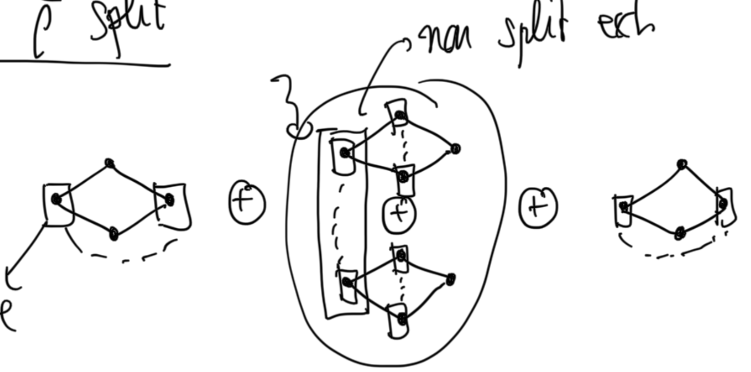
$$\left[\ D_0(\bar\rho) = \bigoplus_{\sigma\in W(\bar\rho)} \underbrace{D_{0,\sigma}(\bar\rho)}_{\hookrightarrow\ \mathrm{soc}\, D_{0,\sigma}(\bar\rho)=\sigma}\ \right]$$

② $D_0(\bar\rho)$ is mult. free

$D_0(\bar\rho)^{I_1} \circlearrowleft I/I_1$ is mult. free and stable under $\chi \mapsto \chi^s$.

Ex: $f=2$, $\bar\rho$ split

$D_0(\bar\rho)$:



$\bullet$ = S.w.

Thm 3 (BP) ① If $\bar\rho$ is irred. or $\bar\rho$ is reducible non split, one cannot write $D_0(\bar\rho)$ as $D \oplus D'$ where $\chi \mapsto \chi^s$ preserves D^{I_1} and D'^{I_1}

② if $\bar\rho$ is red. split, one has:

$$D_0(\bar\rho) = D_{0,0}(\bar\rho) \oplus D_{0,1}(\bar\rho) \oplus \cdots \oplus D_{0,f}(\bar\rho)$$

where $D_{0,i}(\bar\rho) := \bigoplus_{\lg(\sigma)=i} D_{0,\sigma}(\bar\rho)$ and

each $D_{0,i}(\bar\rho)^{I_1}$ is stable under $\underline{\underline{\chi \leftrightarrow \chi^s}}$.

Thm B (Emerton, Gee, Savitt, Hu, Wang, Schraen, Le, Morra (+ BHHMS)) $\leadsto$ $(d>1)$

$d=1$ '' "minimal case"

Same assumption as for Thm A on S.W. + p inert in $F|_{F^+}$ (for simplicity) + some small further technical assumpl. on $\bar r$ and U^v, then there is an integer $d \geqslant 1$ s.t.

$$\boxed{\pi_v[\bar r]^{K_1} = D_0(\bar\rho)^{\oplus d}} \circlearrowleft GL_2(\mathbb{F}_q)$$

rk: If $d=1$, thm B $\Longrightarrow$ weight cycling is $\sigma \leadsto \delta(\sigma) \leadsto \cdots$

③ Diagrams

Definition (Schneider-Stuhler, Paškūnas) A diagram is a triple (D_0, D_1, r) where:

D_0 = smooth repres. of $GL_2(\mathcal{O}_K)K^\times$ over $\mathbb{F}$

D_1 = smooth repres. of $I\begin{pmatrix}0&1\\p&0\end{pmatrix}^{\mathbb{Z}}$ over $\mathbb{F}$

$r : D_1 \longrightarrow D_0$ = $IK^\times$-equiv. morphism

we only consider diagrams where:

- $K^\times$ acts by a character
- $D_0 = D_0^{K_1}$ + is finite dim'l
- $D_1|_{IK^\times} = D_0^{I_1}$ + r = canonical injection $D_0^{I_1} \hookrightarrow D_0 = D_0^{K_1}$.

Not: : $(D_1 \hookrightarrow D_0)$

(ω = mod p cycl

Main example: $(D_1(\bar\rho) \hookrightarrow D_0(\bar\rho))$ where $p \in K^\times$ acts on $D_0(\bar\rho)$ by $(\det(\bar\rho)\omega^{-1})(p)$ and $D_1(\bar\rho) := D_0(\bar\rho)^{I_1}$ endowed with a choice of an action of $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ (possible as stable under $\chi \leftrightarrow \chi^s$).

Thm 4 (Paškūnas, BP) Assume $p > 2$ and let $(D_1 \hookrightarrow D_0)$ be a diagram as above. Then there exists a smooth admissible repr. π of $Gl_2(K)$ over $\mathbb{F}$ such that:

$$(D_1 \hookrightarrow D_0) \underline{\underline{\hookrightarrow}} (\pi^{I_1} \hookrightarrow \pi^{K_1})$$

$$\mathrm{soc}_{Gl_2(O_K)} \pi = \mathrm{soc}_{Gl_2(\mathbb{F}_q)} D_0$$

$$\pi = \langle Gl_2(K). D_0 \rangle .$$

It is NOT true that one can always find π as in Thm 4 with $\pi^{K_1} = D_0$.

It is true for $D_0(\bar{\rho})$ but the proof is global

Thm 5 (BC): Assume $\bar{\rho}$ is reducible split and $(D_1 \hookrightarrow D_0) := (D_{1,\ell}(\bar{\rho}) \hookrightarrow D_{0,\ell}(\bar{\rho}))$ for $\ell \in \{0, \dots, f\}$, or $\bar{\rho}$ is irreducible and $(D_1 \hookrightarrow D_0) = (D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho}))$.
Then any π as in Thm 4 is also irreducible.

Rk: Even if $(\pi^{I_1} \hookrightarrow \pi^{K_1}) = \bigoplus_{\ell} (D_{1,\ell}(\bar{\rho}) \hookrightarrow D_{0,\ell}($
($\bar{\rho}$ red. split)
$\not\Longrightarrow$ π is semi-simple.

Thanks!