# Functor $D$ to $(\varphi,\Gamma)$-modules and a lower bound for $D(\pi)$ (Lecture 4)

$K/\mathbb{Q}_p$ fin.
$\mathbb{F}/\mathbb{F}_p$ fin (large)

## §1 The functor $D_\xi^\vee$

$G := GL_n \quad (n>1)$

$\cup$

$B := \begin{pmatrix} * & * \\ & * \end{pmatrix}, \quad N := \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

$\cup$

$T := \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$

$\xi : \mathbb{G}_m \longrightarrow T$ $\quad [\langle \xi, \alpha \rangle = 1 \ \forall \alpha \text{ simple}]$

$x \longmapsto \begin{pmatrix} x^{n-1} & & & \\ & x^{n-2} & & \\ & & \ddots & \\ & & & x \\ & & & & 1 \end{pmatrix}$

$N_0 := \begin{pmatrix} 1 & \mathcal{O}_K \\ & \ddots & \\ & & 1 \end{pmatrix} \subset N(K)$ cpt. open subgp.

$N_1 := \ker(N_0 \twoheadrightarrow \mathcal{O}_K \xrightarrow{\mathrm{Tr}} \mathbb{Z}_p)$

$g \longmapsto \sum g_{i,i+1}$

Fix $N_0/N_1 \cong \mathbb{Z}_p$.

Note: $\xi(x) N_1 \xi(x)^{-1} \subset N_1 \quad \forall x \in \mathbb{Z}_p \setminus \{0\}$. $\quad (*)$

$\pi$ smooth rep. of $B(K)$ over $\mathbb{F}$

$\Rightarrow \pi^{N_1}$ ------- $N_0/N_1$ ---, so

$\pi^{N_1}$ is a torsion $\mathbb{F}[N_0/N_1]$-module

$\qquad \| $

$\mathbb{F}[\![\mathbb{Z}_p]\!] = \mathbb{F}[\![X]\!]$

$X := [1]-1$.

By $(*)$:

$\Gamma := \mathbb{Z}_p^\times \curvearrowright \pi^{N_1}$ via $\xi$,

$F \curvearrowright \pi^{N_1}$ via

$$F(v) = \sum_{n \in N_1/\xi(p) N_1 \xi(p)^{-1}} n \cdot \xi(p) v$$

**Lemma:** On $\mathbb{F}[\![X]\!]$-module $\pi^{N_1}$ have:

(i) $F \circ X = X^p \circ F$

(ii) $\gamma \circ X = ((1+X)^\gamma - 1) \circ \gamma \quad \forall \gamma \in \Gamma = \mathbb{Z}_p^\times$

(iii) $\gamma \circ F = F \circ \gamma$

(cf. lect. 2)

let

$$\mathcal{M} := \{ \text{torsion } \mathbb{F}[\![X]\!]\text{-modules } M \text{ + actions of } F \text{ and } \Gamma \text{ as in Lemma} \} \ni \pi^{N_1}$$

$\cup$

$$\mathcal{M}_{fin} := \{ M \in \mathcal{M} : M \text{ f.g. as } \mathbb{F}[\![X]\!][F]\text{-mod.} + M \underline{\text{adm}}, \text{ i.e. } \dim_{\mathbb{F}} M[X] < \infty \}.$$

**Key Prop.** (Colmez): If $M \in \mathcal{M}_{fin}$, then $M^\vee[1/X]$ becomes a f.d. $(\varphi, \Gamma)$-module over $\mathbb{F}((X))$.

Pf: $M^\vee := \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ $\left( = \varprojlim_i M[X^i]^\vee \right)$

cpt. $\mathbb{F}[\![X]\!]$-module

$M[X]$ f.d. $\Leftrightarrow$ $M^\vee / X M^\vee$ f.d.

$\Leftrightarrow$ $M^\vee$ is f.g. as $\mathbb{F}[\![X]\!]$-mod.

For $f \in M^\vee$, let

$$\left.\begin{aligned} Xf &:= f \circ X \\ Ff &:= f \circ F \end{aligned}\right\} \Rightarrow X \circ F = F \circ X^p \text{ on } M^\vee$$

$$\gamma f := f \circ \gamma^{-1}$$

"wrong!"

Idea:

Consider

$$\mathrm{id} \otimes F : \mathbb{F}[\![X]\!] \otimes_{\varphi, \mathbb{F}[\![X]\!]} M \longrightarrow M$$

f.d. coker

$$\Rightarrow (\mathrm{id} \otimes F)^\vee : M^\vee \longrightarrow (\mathbb{F}[\![X]\!] \otimes_\varphi M)^\vee \cong \mathbb{F}[\![X]\!] \otimes_\varphi (M^\vee)$$

$$f \longmapsto \sum_{i=0}^{p-1} (1+X)^i \otimes_\varphi f((1+X)^{-i} \otimes_\varphi (-))$$

Localise at $X$:

$M^\vee[1/X] \hookrightarrow \mathbb{F}((X)) \otimes_\varphi (M^\vee[1/X])$ becomes an isom.

Let $\varphi$ on $M^\vee[1/X]$ be the inverse of this isom. $\square$

Def. (Breuil)

$$D_\xi^\vee(\pi) := \varprojlim_{\substack{M \subset \pi^{N_1} \\ \text{in } \mathcal{M}_{fin}}} M^\vee[1/X] \quad \text{pro-}(\varphi,\Gamma)\text{-mod}$$

[filtered w.r.t. inclusion, surj. trans. maps]

$$V_{GL_n}(\pi) := \varinjlim_{\substack{M \subset \pi^{N_1} \\ \text{in } \mathcal{M}_{fin}}} \underbrace{\mathbb{V}(M^\vee[1/X])^\vee \otimes \delta_{GL_n}}_{\text{f.d. rep. of } G_{\mathbb{Q}_p}}$$

where $\delta_{GL_n} := \omega^{[K:\mathbb{Q}_p]\cdot \sum_{i=1}^{n-1} i^2}$.

Rk: (i) For $GL_2(\mathbb{Q}_p)$ $D_\xi^\vee$ coincides with Colmez fr. [up to normal$^n$]

(ii) $V_{GL_n}$ is left exact + compat. with parab. ind$^n$.

$\boxed{\text{From now: } n=2, \ K/\mathbb{Q}_p \text{ unram. deg. } f}$

$$N_0 = \begin{pmatrix} 1 & \mathcal{O}_K \\ & 1 \end{pmatrix}$$
$$\triangledown$$
$$N_1 = \begin{pmatrix} 1 & \mathcal{O}_K^{Tr=0} \\ & 1 \end{pmatrix}$$
$$\xi(x) = \begin{pmatrix} x & \\ & 1 \end{pmatrix}$$

## §2 Lower bound for $D_\xi^\vee(\pi(\bar{r}))$

Fix

$$\bar{\rho} : G_{\mathbb{Q}_{p^f}} \longrightarrow GL_2(\mathbb{F}) \quad \text{s.t.}$$

$$\bar{\rho}|_{I_{\mathbb{Q}_{p^f}}} \cong \begin{cases} \omega_f^{\sum_0^{f-1}(r_j+1)p^j} \oplus 1 & \bar{\rho} \text{ split reducible} \\ \omega_{2f}^{\sum (r_j+1)p^j} \oplus \omega_{2f}^{p^f\sum(r_j+1)p^j} & \bar{\rho} \text{ irred.} \end{cases}$$

[up to twist]

where

$$2f \le r_j \le p-2-2f \quad (\text{"genericity"})$$

[can weaken slightly]

$$\bar{\rho} \xrightarrow[\text{BP + Dotto-Le}]{} \text{diagram } (D_1(\bar{\rho}) \hookrightarrow D_0(\bar{\rho})) =: D(\bar{\rho})$$

[BP: family]

Thm: Assume $\pi$ (adm.) smooth rep. of $GL_2(\mathbb{Q}_{p^f})$ over $\mathbb{F}$ and $r \geq 1$ s.t.

$$D(\bar{\rho})^{\oplus r} \cong (\pi^{I_1} \hookrightarrow \pi^{K_1}) \quad \text{as diagrams,}$$

then

$$\left(\mathrm{Ind}_{G_{\mathbb{Q}_{p^f}}}^{\otimes G_{\mathbb{Q}_p}} \bar{\rho}\right)^{\oplus r} \hookrightarrow V_{GL_2}(\pi)$$

$$(\Rightarrow \dim D_\xi^\vee(\pi) \geq 2^f r)$$

**Rough strategy:** Assume $r=1$.

**Need:** $M \subset \pi^{N_1}$ s.t. $M$ is f.g. $\mathbb{F}[\![X]\!][F]$-mod., $M[X]$ f.d., $\Gamma$-stable s.t. $\dim_{\mathbb{F}(\!(X)\!)}(M^\vee[1/X]) = 2f$

**Basic Lemma:**

If $v_1, \dots, v_n \in \pi^{N_1}[X] = \pi^{N_0}$ are $\Gamma$-evecs that are $\mathbb{F}$-lin. indep. and

$$X^{s_i} \cdot F(v_i) \in \mathbb{F}[\![X]\!]^\times \cdot v_{i+1} \quad \forall i$$

(some $s_i > 0$), then $M := \bigoplus_{i=1}^n \mathbb{F}[\![X]\!][F] \cdot v_i \in \mathcal{M}_{fin}$

and $\dim M^\vee[1/X] = n$.

**Improved lemma:**

If $v_1, \dots, v_n \in \pi^{N_1}$ are $\Gamma$-evecs. s.t.

$X^d v_1, \dots, X^d v_n \in \pi^{N_1}[X]$ and $\mathbb{F}$-lin. indep.

(some $d \geq 0$) and

$$X^{s_i} \cdot F(v_i) \in \mathbb{F}[\![X]\!]^\times \cdot v_{i+1} \quad \forall i,$$

(some $s_i > 0$), then same conclusion.

**Pf:** say $X^{s_i} F(v_i) = v_{i+1}$ $\forall i$.

Action of $X$:

$$(0 \leftarrowtail) X^d v_i \leftarrowtail X^{d-1} v_i \leftarrowtail \cdots \leftarrowtail v_i \leftarrowtail X^{s_{i-1}-1} F(v_{i-1})$$

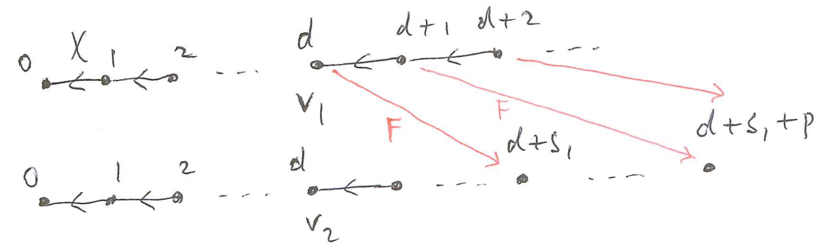
$$\cdots \leftarrowtail F(v_{i-1}) \leftarrowtail X^{p s_{i-2}-1} F^2(v_{i-2}) \leftarrowtail \cdots$$

As $(X^d v_i)_{i=1}^n$ lin. indep., the vectors above $(1 \leq i \leq n)$ are lin. indep., hence an $\mathbb{F}$-basis of $M$.

Deduce: $X : M \to M$ is surj. and $\dim M[X] = n$

$\Rightarrow M^\vee$ is free of rk. $n$ over $\mathbb{F}[\![X]\!]$.

Picture:




□

Let $x_i := X^d v_i \in M[X]$

$f_i \in M^\vee$ the "dual basis" over $\mathbb{F}[\![X]\!]$

(i.e. $f_i(x_i) = 1$ and zero on all other basis vectors)

Use Colmez prop$^n$ to compute $M^\vee[{}^1/X]$:

$$\begin{cases} \varphi(f_i) = (1+X)^{p-1} \cdot X^{s_i-(d+1)(p-1)} f_{i+1} \\ \gamma(f_i) \in \bar{\gamma}^{-d} \cdot \chi_i(\gamma)^{-1}(1+X\mathbb{F}[\![X]\!]) f_i, \end{cases}$$

where $\gamma(v_i) = \chi_i(\gamma) v_i$.

To calculate Galois rep$^n$:

**Lemma:** Suppose the $(\varphi, \Gamma)$-module $D$ has basis $e_1, \dots, e_n$ s.t.

$$\begin{cases} \varphi(e_i) \in \mathbb{F}[\![X]\!]^\times \cdot X^{s_i} \cdot e_{i+1} \\ \gamma(e_i) \in \bar{\gamma}^{a_i}(1+X\mathbb{F}[\![X]\!]) \cdot e_i \end{cases} \qquad (a_i, s_i \in \mathbb{Z})$$

then

$$\mathbb{V}(D)^\vee \otimes \underbrace{\delta_{GL_2}}_{\omega^f}|_{I_{\mathbb{Q}_p}} = \omega^{f-a_1} \otimes \mathrm{Ind}_{G_{\mathbb{Q}_{p^n}}}^{G_{\mathbb{Q}_p}}(\omega_n^s)|_{I_{\mathbb{Q}_p}}$$

where

$$s := \frac{p^{n-1}s_1 + p^{n-2}s_2 + \dots + s_n}{p-1} \in \mathbb{Z}.$$

## §3 Proof of Thm ($f=2$)

⚠ $K$ not a field!

$$\begin{array}{ccc} K := GL_2(\mathbb{Z}_{p^f}) & \supset & I \\ \cup & & \cup \\ K_1 & \subset & I_1 \end{array}$$

Take $\bar{\rho}$ split reducible:

$$\bar{\rho}|_{I_{\mathbb{Q}_{p^2}}} \cong \omega_2^{(r_0+1)+p(r_1+1)} \oplus 1 \quad (+\text{generic!})$$

<u>Hard case</u> (2-cycle): (ignore twists)

$$\sigma = (r_0+1,\ p-2-r_1) \rightleftarrows \delta(\sigma) = (p-2-r_0,\ r_1+1) \in W(\bar{\rho}).$$

Pick $x_\sigma \in \sigma^{N_0}\setminus\{0\}$, $I$-evals. $\chi_\sigma$

Weight cycling (Lecture 3):

$$\operatorname{Ind}_I^K(\chi_\sigma^s) \twoheadrightarrow \underset{\displaystyle \overset{\|}{Q}}{\langle K\cdot \begin{pmatrix} & 1 \\ p & \end{pmatrix} x_\sigma\rangle} \hookrightarrow \pi$$

(image lands in $\pi^{K_1} = D_0(\bar{\rho})$!)

$$\operatorname{Ind}_I^K(\chi_\sigma^s)$$

$$\|$$

$$\left(\begin{array}{ccc} & (p-2-r_0,\ r_1+1) & \\ / & & \backslash \\ (r_0, r_1) & & (p-3-r_0,\ p-3-r_1) \\ \backslash & & / \\ & (r_0+1,\ p-2-r_1) & \end{array}\right) \twoheadrightarrow \underset{\displaystyle \overset{\|}{\delta(\sigma)}}{\overset{\displaystyle Q}{\overset{\|}{(p-2-r_0,\ r_1+1)}}}$$

$$\underset{\displaystyle K\text{-socle } \sigma}{\|}$$

(by "mult. one" property of $D_0(\bar{\rho})$, cf. Lecture 3)

circled: wts. in $W(\bar{\rho})$

<u>Def.</u>: $H := \left\{ \begin{pmatrix} [x] & \\ & [y] \end{pmatrix} : x, y \in \mathbb{F}_{p^f}^\times \right\}$

$$\alpha : H \longrightarrow \mathbb{F}^\times$$
$$\begin{pmatrix} [x] & \\ & [y] \end{pmatrix} \longmapsto xy^{-1}$$

$$Y_j := \sum_{a \in \mathbb{F}_{p^f}^\times} a^{-p^j} \begin{pmatrix} 1 & [a] \\ & 1 \end{pmatrix} \in \mathbb{F}[\![N_0]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$$

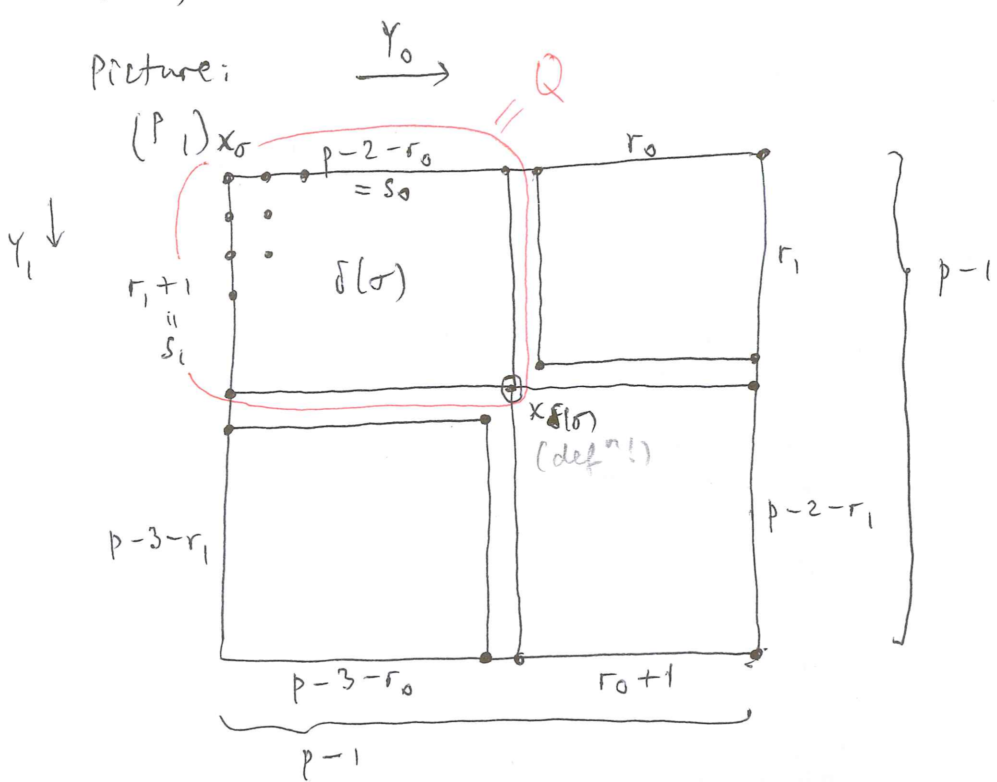
<u>Rk:</u> $h \circ Y_j \circ h^{-1} = \alpha(h)^{p^j} \cdot Y_j \quad \forall h \in H$

**Def.:** $Y := \sum_{a \in \mathbb{F}_p^\times} a^{-1} \begin{pmatrix} 1 & [a] \\ & 1 \end{pmatrix} \in \mathbb{F}[\![\mathbb{Z}_p]\!] = \mathbb{F}[\![Y]\!]$

**Lemma (BP):** If $\mathrm{Ind}_I^K(\chi_\sigma^s) \twoheadrightarrow Q$ is a proper quotient, then $Q$ is a cyclic $\mathbb{F}[\![N_0]\!]$-module generated by $\underbrace{\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ p & \end{pmatrix} x_\sigma}_{= \begin{pmatrix} p & \\ & 1 \end{pmatrix} x_\sigma}$
$= \mathbb{F}[\![Y_0, Y_1]\!]$.
and $H$-eigenbasis $Y_0^{i_0} Y_1^{i_1} \begin{pmatrix} p & \\ & 1 \end{pmatrix} x_\sigma$
(+ multiplicity free for $H$)

Here, $Q = \delta(\sigma) = (p-2-r_0, r_1+1) =: (s_0, s_1)$

Picture:




(circled: $\underline{Y}$-torsion $\Leftrightarrow$ $N_0$-inv. in $Q$)

shorthand!

$\Rightarrow$ If $\underline{Y}^{\underline{i}} \begin{pmatrix} p & \\ & 1 \end{pmatrix} x_\sigma \neq 0$, then $i_0 + i_1 \leq s_0 + s_1$ and if equality holds, then $\underline{i} = (s_0, s_1)$

**Recall:** $F(v) = \sum_{N_1/N_1^p} n_1 \cdot \begin{pmatrix} p & \\ & 1 \end{pmatrix} v$ for $v \in \pi^{N_1}$

**Lemma:** For any $0 \leq j_0 \leq f-1$ we have

$$\sum_{N_1/N_1^p} n_1 = (-1)^{f-1} \prod_{j \neq j_0} (Y_j - Y_{j_0})^{p-1} + (\deg \geq f(p-1))$$

$$\in \mathbb{F}[\![N_0/N_1^p]\!] = \mathbb{F}[\![N_0]\!]/((Y_i - Y_j)^p : i \neq j)$$

**Lemma:**
$$\begin{array}{ccc} \vartheta_K & & \\ \| & & \\ \mathbb{F}[\![N_0]\!] & \overset{Tr}{\twoheadrightarrow} & \mathbb{F}[\![N_0/N_1]\!] = \mathbb{F}[\![\mathbb{Z}_p]\!] \\ Y_j & \longmapsto & Y + (\deg \geq p) \end{array}$$

By above,

$\underline{Y}^{\underline{i}} \begin{pmatrix} p & \\ & 1 \end{pmatrix} x_\sigma = 0$ if $i_0 + i_1 > s_0 + s_1 = p-1-r_0+r_1$

**Problem:** If $r_0 > r_1$, then

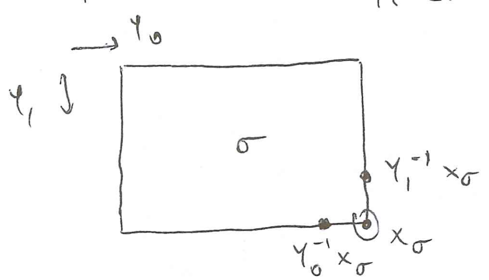
$$F(x_\sigma) = \left(-(Y_0-Y_1)^{p-1} + (\deg. \geq 2p-2)\right)\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) x_\sigma = 0 \ !$$

**Solution:**

$\sigma^{N_1}$ is cyclic as $\underset{\mathbb{F}[\![X]\!]}{\mathbb{F}[\![Y]\!]}$-mod. of dim $= \min(r_0+1, p-2-r_1)+1$.

Take

$$v_\sigma := \underset{\text{I-evec}}{\overset{\in \sigma^{N_1}}{Y^{-1} x_\sigma}} = \underbrace{Y_0^{-1} x_\sigma + Y_1^{-1} x_\sigma}_{\text{H-evecs}}$$

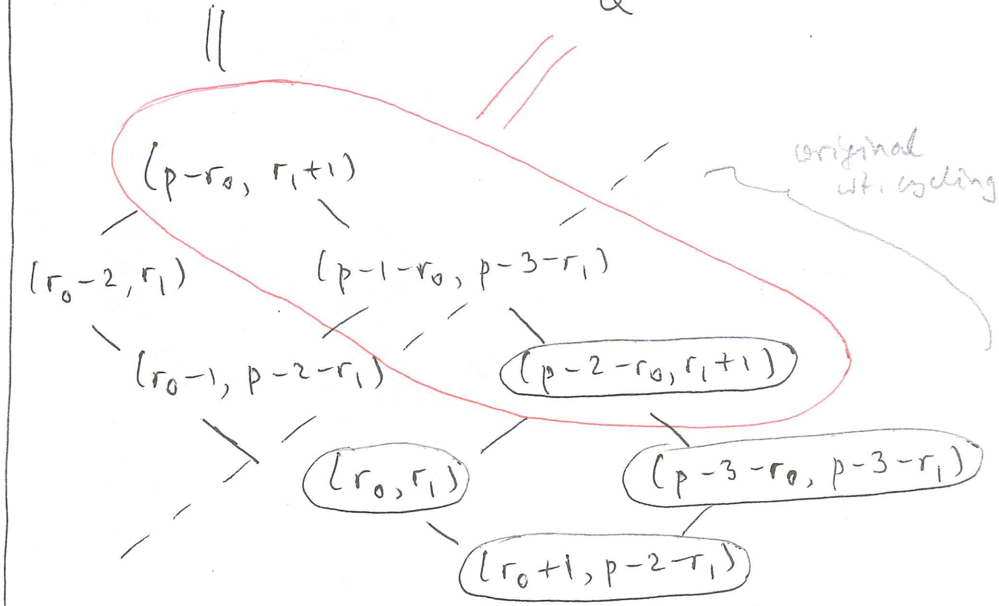



To understand

$$F(v_\sigma) = \sum_{N_1/N_1^p} n_1 \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) v_\sigma = \sum_{j=0}^{1} \sum_{N_1/N_1^p} n_1 \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) Y_j^{-1} x_\sigma$$

use "weight cycling": start from $Y_0^{-1} x_\sigma$ (not I-evec)

$$\mathrm{Ind}_I^K \begin{pmatrix} \chi_\sigma^s \cdot \alpha \\ | \\ \chi_\sigma^s \end{pmatrix} \twoheadrightarrow \underset{= Q}{\left\langle K \cdot \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) Y_0^{-1} x_\sigma \right\rangle} \hookrightarrow \pi$$




($\Rightarrow$ need stronger genericity!)

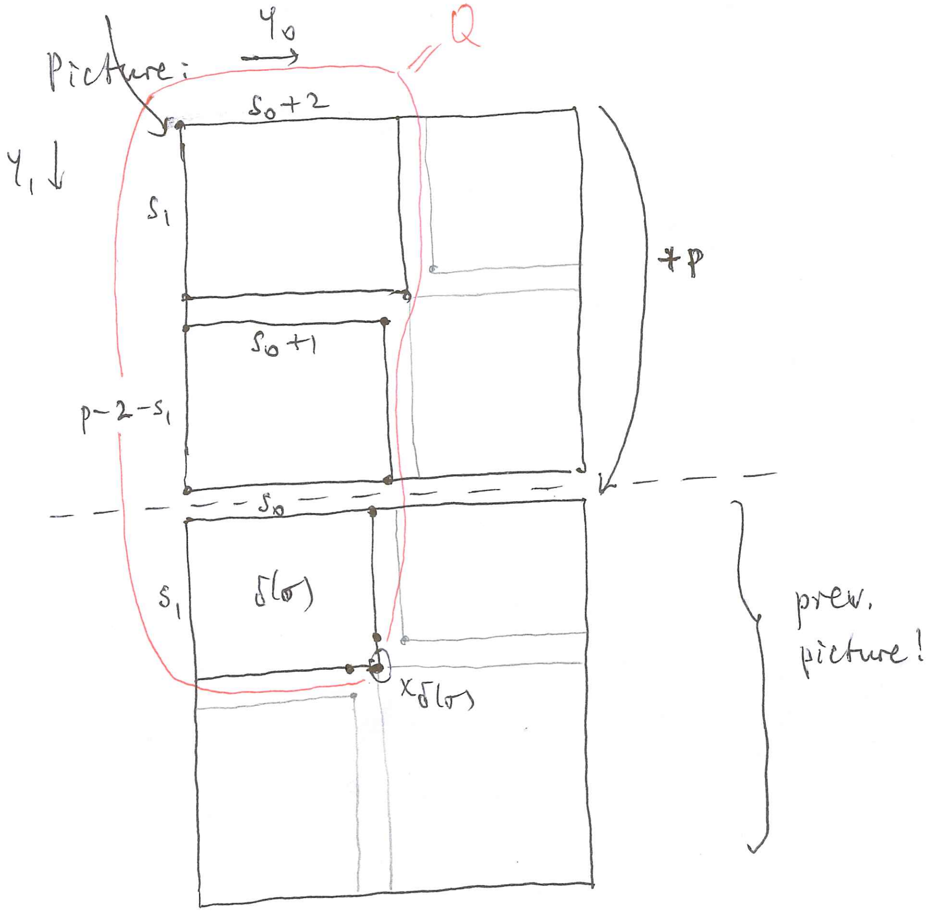
**Lemma:** $Y_i^p \circ \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \circ Y_{i+1}$.

BP lemma (twice) $\Rightarrow$ $Q$ is a cyclic $\mathbb{F}[N_0]$-mod generated by $\begin{pmatrix} p & \\ & 1 \end{pmatrix} Y_0^{-1} x_\sigma$

(relations?)

$\begin{pmatrix} p & \\ & 1 \end{pmatrix} Y_0^{-1} x_\sigma$

Picture:




(not mult. free for $H$)

Lemma: Assume

$$\underline{Y}^{\underline{i}} \begin{pmatrix} p & \\ & 1 \end{pmatrix} Y_0^{-1} x_\sigma \neq 0 \Rightarrow i_0 + i_1 \le s_0 + s_1 + p.$$

If $i_0 + i_1 = s_0 + s_1 + p$, then $\underline{i} = (s_0, s_1 + p)$.

If $i_0 + i_1 = s_0 + s_1 + p - 1$, then

$$\underline{Y}^{\underline{i}} \begin{pmatrix} p & \\ & 1 \end{pmatrix} Y_0^{-1} x_\sigma \in \langle Y_0^{-1} x_{\delta(\sigma)}, Y_1^{-1} x_{\delta(\sigma)} \rangle_{\mathbb{F}}$$

Pf: BP lemma + key relation $Y_0^{s_0+3} \begin{pmatrix} p & \\ & 1 \end{pmatrix} Y_0^{-1} x_\sigma = 0$. $\square$

Deduce:

$$Y_0^{s_0} Y_1^{s_1} (Y_0 - Y_1)^{p-1} \begin{pmatrix} p & \\ & 1 \end{pmatrix} Y_j^{-1} x_\sigma \in \langle Y_0^{-1} x_{\delta(\sigma)}, Y_1^{-1} x_{\delta(\sigma)} \rangle_{\mathbb{F}}.$$

for $j = 0$ and similarly for $j = 1$.

$$\Rightarrow \boxed{Y^{s_0+s_1} F(Y^{-1} x_\sigma) \in \mathbb{F}^\times \cdot Y^{-1} x_{\delta(\sigma)}}$$

As $Y \in -X(1 + X\mathbb{F}[[X]])$, conclude

$$\boxed{X^{s_0+s_1} F(v_\sigma) \in \mathbb{F}[[X]]^\times \cdot v_{\delta(\sigma)}}$$

Iterate (2-cycle) $\longrightarrow$ wt. cycling constant!

(cf. end of Lecture 3)

<u>General formulas ($f \geq 1$):</u>

$$\sigma = (t_0, t_1, \ldots) \otimes \ldots \quad \in W(\bar{\rho})$$

$$\delta(\sigma) = (s_0, s_1, \ldots) \otimes \ldots$$

$$J^{max}(\sigma) := \{ j : s_j + t_j \approx p \}.$$

$$m := |J^{max}(\sigma)|$$

$$x_{\delta(\sigma)} := \prod_{j \in J^{max}(\sigma)} Y_j^{s_j} \cdot \prod_{j \notin J^{max}(\sigma)} Y_j^{p-1} \cdot \binom{p}{1} x_\sigma \quad \in \delta(\sigma)^{N_0} \setminus \{0\}$$

and

$$\begin{cases} Y^{\sum_{J^{max}(\sigma)} s_j} \cdot F(Y^{1-m} \cdot x_\sigma) = (-1)^{f-1} \cdot Y^{1-m} x_{\delta(\sigma)} & (\text{if } m > 0) \\ Y^{p-1} F(x_\sigma) = (-1)^{f-1} x_{\delta(\sigma)} & (\text{if } m = 0) \end{cases}$$

$\boxed{\text{Summary: } f=2,\ \bar{\rho} \text{ split reducible}}$ $\qquad \bar{\rho}|_{I_{\mathbb{Q}_{p^2}}} \cong \omega_2^{(r_0+1)+p(r_1+1)} \oplus \mathbb{1}$

| Serre wts. of $\bar{\rho}$ | $J^{max}(\sigma_i)$ | $\sum_{J^{max}(\sigma_i)} s_j$ | $\Gamma$-evals. of $\sigma_i^{N_0}$ |
|---|---|---|---|
| $\delta \circlearrowleft \sigma_1 = (r_0, r_1)$ | $\emptyset$ | | $\gamma \mapsto \bar{\gamma}^{r_0+r_1}$ |
| $\delta \circlearrowleft \sigma_2 = (p-3-r_0, p-3-r_1) \otimes \det^{(r_0+1)+p(r_1+1)}$ | $\emptyset$ | | $\gamma \mapsto \bar{\gamma}^{-2}$ |
| $\sigma_3 = (r_0+1, p-2-r_1) \otimes \det^{-1+p(r_1+1)}$ | $\{0,1\}$ | $p-1-r_0+r_1$ | $\gamma \mapsto \bar{\gamma}^{-r_0}$ |
| $\sigma_4 = (p-2-r_0, r_1+1) \otimes \det^{(r_0+1)-p}$ | $\{0,1\}$ | $p-1+r_0-r_1$ | |

$\Rightarrow$

$$\boxed{\begin{array}{ll} \varphi(f_1) \sim \lambda_1 f_1, & \gamma(f_1) \sim \bar{\gamma}^{-r_0-r_1} f_1 \\ \varphi(f_2) \sim \lambda_2 f_2, & \gamma(f_2) \sim \bar{\gamma}^{2} f_2 \\ \varphi(f_3) \sim \lambda_3 X^{-r_0+r_1-p+1} f_4, & \gamma(f_3) \sim \bar{\gamma}^{-r_0} f_3 \\ \varphi(f_4) \sim \lambda_4 X^{r_0-r_1-p+1} f_3 & \end{array}}$$

$\sim$ means up to $1 + X\mathbb{F}[[X]]$, $\lambda_i \in \mathbb{F}^\times$

$(\varphi, \Gamma)$-module $M$ over $\mathbb{F}((X))$ with basis $(f_i)$

$$\Rightarrow \quad \mathbb{V}(M)^\vee \otimes \delta_{\mathrm{GL}_2}|_{I_{\mathbb{Q}_p}} \cong \omega^{(r_0+1)+(r_1+1)} \oplus \mathbb{1} \oplus \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}}\left(\omega_2^{(r_1+1)+p(r_0+1)}\right)\Big|_{I_{\mathbb{Q}_p}}$$

$$\cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{\otimes G_{\mathbb{Q}_p}}(\bar{\rho})|_{I_{\mathbb{Q}_p}}.$$

$\boxed{\text{Summary: } f=2,\ \bar\rho \text{ irred}}$ $\qquad \bar\rho|_{I_{\mathbb{Q}_{p^2}}} \cong \omega_4^{(r_0+1)+p(r_1+1)} \oplus \omega_4^{p^2(\cdots)}$

| Serre wts. of $\bar\rho$ | $J^{max}(\sigma_i)$ | $\sum_{J^{max}(\sigma_i)} s_j$ | $\Gamma$-evals. of $\sigma_i^{N_0}$ |
|---|---|---|---|
| $\sigma_1 = (r_0, r_1)$ | $\{1\}$ | $p-2-r_1$ | $\gamma \mapsto \bar\gamma^{\,r_0+r_1}$ |
| $\sigma_2 = (r_0-1,\ p-2-r_1) \otimes \det^{p(r_1+1)}$ | $\{0\}$ | $p-1-r_0$ | |
| $\sigma_3 = (p-1-r_0,\ p-3-r_1) \otimes \det^{r_0+p(r_1+1)}$ | $\{1\}$ | $r_1+1$ | |
| $\sigma_4 = (p-2-r_0,\ r_1+1) \otimes \det^{(r_0+1)-p}$ | $\{0\}$ | $r_0$ | |

$$\Rightarrow \boxed{\begin{aligned} \varphi(f_1) &\sim \lambda_1 \cdot X^{-1-r_1} f_2, \qquad \gamma(f_1) \sim \bar\gamma^{\,-r_0-r_1} \cdot f_1 \\ \varphi(f_2) &\sim \lambda_2 \cdot X^{-r_0} f_3 \\ \varphi(f_3) &\sim \lambda_3 \cdot X^{r_1-p+2} f_4 \\ \varphi(f_4) &\sim \lambda_4 \cdot X^{r_0-p+1} f_1 \end{aligned}}$$

where $\sim$ means up to $1 + X\mathbb{F}[\![X]\!]$, $\lambda_i \in \mathbb{F}^\times$.

$(\varphi, \Gamma)$-module $M$ over $\mathbb{F}((X))$ with basis $(f_i)$

$$\Rightarrow \mathbb{V}(M)^\vee \otimes \delta_{GL_2}|_{I_{\mathbb{Q}_p}} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_p}} \left( \omega_4^{(r_0+1)(p^2+p^3)+(r_1+1)(1+p^3)} \right)\Big|_{I_{\mathbb{Q}_p}}$$

$$\cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{\otimes G_{\mathbb{Q}_p}} (\bar\rho)\big|_{I_{\mathbb{Q}_p}}.$$